Mathematical Presentation of Input-Output Analysis

$$\overbrace{z_{11} + z_{12} \quad + \; F_1}^{\text{需要}} \; = \; \overbrace{X_1}^{\text{供給}} \leftarrow \text{財1の需給バランス式}$$

$$\underbrace{z_{21} + z_{22}}_{\text{中間需要}} \quad \underbrace{+ \; F_2}_{\text{最終需要}} \; = \; \underbrace{X_2}_{\text{総生産}} \leftarrow \text{財2の需給バランス式}$$

ただし:

z_{11} = 財1に対する産業1の中間需要

z_{12} = 財1に対する産業2の中間需要

F_1 = 財1への最終需要

X_1 = 財1の総生産額、

財2についても同様。

	Sector 1	Sector 2	Final Demand	Total Output
Sector 1	10 z_{11}	20 z_{12}	70 F_1	100 X_1
Sector 2	40 z_{21}	100 z_{22}	60 F_2	200 X_2
Value Added	50 V_1	80 V_2		
Total Output	100 X_1	200 X_2		

ただし:

z_{11} = 財1に対する産業1の中間需要

z_{12} = 財1に対する産業2の中間需要

F_1 = 財1への最終需要

X_1 = 財1の総生産額、

財2についても同様。

$$a_{11} X_1 + a_{12} X_2 + F_1 = X_1$$

$$a_{21} X_1 + a_{22} X_2 + F_2 = X_2$$

定義上、$a_{ij} = z_{ij} / X_j$, したがって

$a_{11} X_1 = z_{11}$,　$a_{12} X_2 = z_{12}$,　$a_{21} X_1 = z_{21}$,　$a_{22} X_2 = z_{22}$

ただし:

a_{11} =産業1の財1に対する投入係数
a_{12} =産業2の財1に対する投入係数
a_{21} =産業1の財2に対する投入係数
a_{22} =産業2の財2に対する投入係数

$$(1 - a_{11}) X_1 - a_{12} X_2 = F_1$$

$$- a_{21} X_1 + (1 - a_{22}) X_2 = F_2$$

$$X_1 = \frac{1 - a_{22}}{(1 - a_{11})(1 - a_{22}) - a_{12} a_{21}} F_1 + \frac{a_{12}}{(1 - a_{11})(1 - a_{22}) - a_{12} a_{21}} F_2$$

Leontief multiplier

$$X_2 = \frac{a_{21}}{(1 - a_{11})(1 - a_{22}) - a_{12} a_{21}} F_1 + \frac{1 - a_{11}}{(1 - a_{11})(1 - a_{22}) - a_{12} a_{21}} F_2$$

$$a_{11} X_1 + a_{12} X_2 + F_1 = X_1$$

$$a_{21} X_1 + a_{22} X_2 + F_2 = X_2$$

行列への変換

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathrm{x} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

< Matrix presentation of a general case>

財1 $a_{11} \cdot X_1 + a_{12} \cdot X_2 + a_{13} \cdot X_3 + \dots + a_{1n} \cdot X_n + F_1 = X_1$

財2 $a_{21} \cdot X_1 + a_{22} \cdot X_2 + a_{23} \cdot X_3 + \dots + a_{2n} \cdot X_n + F_2 = X_2$

財3 $a_{31} \cdot X_1 + a_{32} \cdot X_2 + a_{33} \cdot X_3 + \dots + a_{3n} \cdot X_n + F_3 = X_3$

$\vdots$

財 n $a_{n1} \cdot X_1 + a_{n2} \cdot X_2 + a_{n3} \cdot X_3 + \dots + a_{nn} \cdot X_n + F_n = X_n$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix}$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Identity matrix:

with 1 in the diagonal elements and 0 elsewhere

$$
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n
\end{pmatrix}
+
\begin{pmatrix}
F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n
\end{pmatrix}
=
\begin{pmatrix}
X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n
\end{pmatrix}
$$

$$
\underset{(n \times n)}{\mathbf{A}} \cdot \underset{(n \times 1)}{\mathbf{X}} + \underset{(n \times 1)}{\mathbf{F}} = \underset{(n \times 1)}{\mathbf{X}}
$$

$$
(\mathbf{I} - \mathbf{A}) \cdot \mathbf{X} = \mathbf{F}
$$

$$
\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{F}
$$

$$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{F}$$

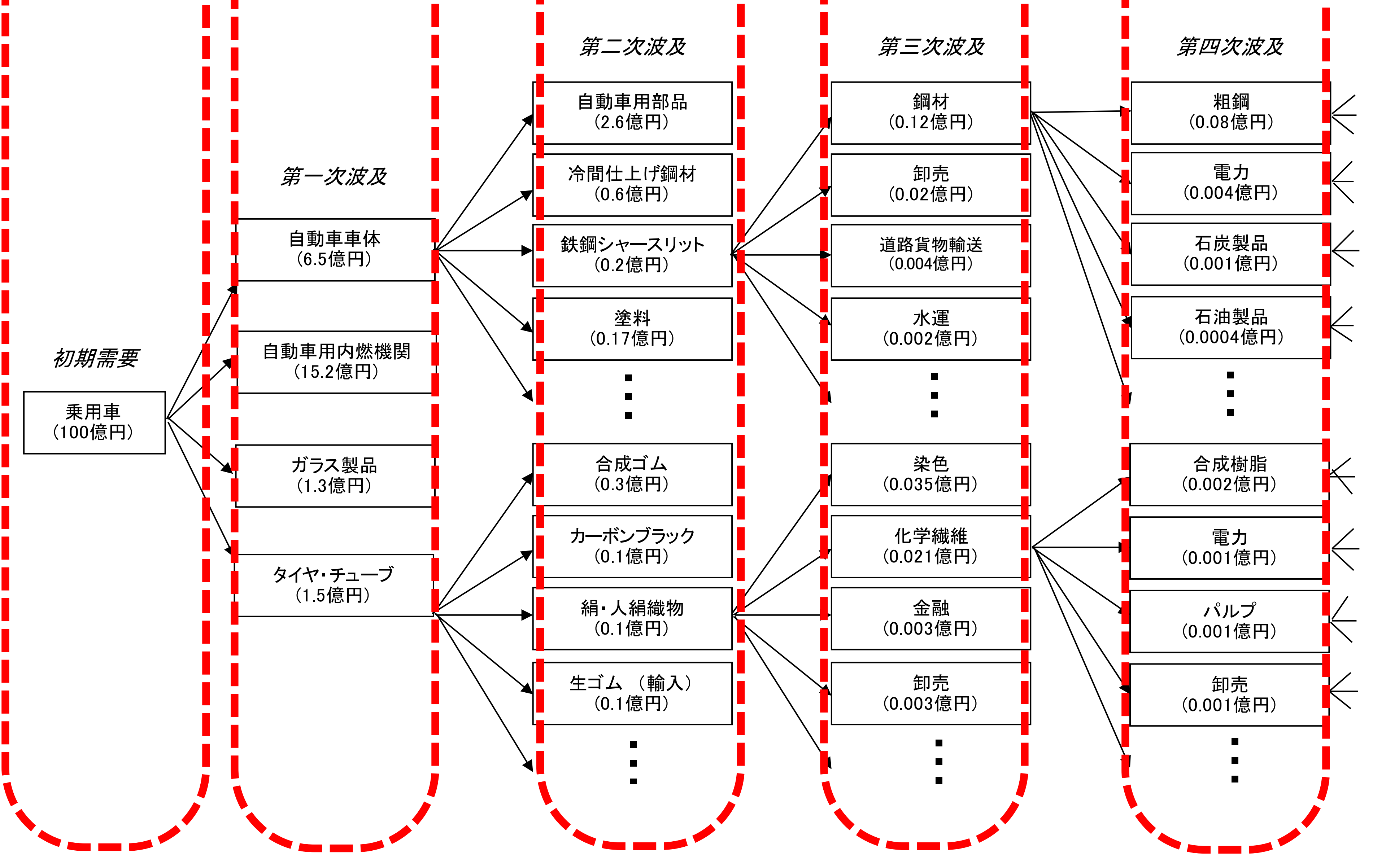
$$= (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \ldots) \cdot \mathbf{F}$$

$$= \underbrace{\mathbf{I} \cdot \mathbf{F}}_{\text{初期需要}} + \underbrace{\mathbf{A} \cdot \mathbf{F}}_{\text{１次波及}} + \underbrace{\mathbf{A}^2 \cdot \mathbf{F}}_{\text{２次波及}} + \cdots$$

→ 生産波及効果の総額を

生産工程ごとに分解

$$L = (I - A)^{-1} = I + A + A^2 + A^3 + A^4 \cdots$$



付加価値貿易 $\boldsymbol{v}(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{F}$

$\boldsymbol{v}$: 付加価値係数

雇用モデル $\boldsymbol{e}(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{F}$

$\boldsymbol{e}$: 雇用係数

炭素フットプリント $\boldsymbol{c}(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{F}$

$\boldsymbol{c}$: 雇用係数

平均波及回数
Average Propagation Length (APL)

ある産業を起点とする生産波及が、
他の産業へ行きつくまでに経る
生産工程数の加重平均

→ サプライチェーンの細分化
（フラグメンテーション）レベル

Average Propagation Lengths Definition

$$v_{ij} = \sum_{k=1}^{\infty} k \left(\left[\mathbf{A}^{k}\right]_{ij} \Big/ \sum_{k=1}^{\infty} \left[\mathbf{A}^{k}\right]_{ij} \right)$$

$[\mathbf{A}^{k}]_{ij}$: the element of i^{th} row and j^{th} column of an input coefficient matrix of k^{th} power

k : a number of production stages along the path

$$v_{ij} = 1 * \underbrace{\frac{\mathbf{A}_{ij}}{\left(l_{ij} - \delta_{ij}\right)}}_{\text{Share of 1-step paths}} + 2 * \underbrace{\frac{\left[\mathbf{A}^{2}\right]_{ij}}{\left(l_{ij} - \delta_{ij}\right)}}_{\text{Share of 2-step paths}} + 3 * \underbrace{\frac{\left[\mathbf{A}^{3}\right]_{ij}}{\left(l_{ij} - \delta_{ij}\right)}}_{\text{Share of 3-step paths}} + \dots$$

The number of stages

l_{ij} : Leontief inverse coefficient

δ_{ij} : Kronecker delta ($\delta_{ij} = 1$ if i=j and $\delta_{ij} = 0$ otherwise)

$v_{ij} = 0$ when $(l_{ij} - \delta_{ij}) = 0$

$$v_{ij} = 1 * \frac{\mathbf{A}_{ij}}{(l_{ij} - \delta_{ij})} + 2 * \frac{\left[\mathbf{A}^2\right]_{ij}}{(l_{ij} - \delta_{ij})} + 3 * \frac{\left[\mathbf{A}^3\right]_{ij}}{(l_{ij} - \delta_{ij})} + \cdots$$

Share of 1-step paths

Share of 2-step paths

Share of 3-step paths

The number of stages

<u>Weighted average</u> **of the number of production stages**

→ Level of technological fragmentation

$$[\mathbf{A}^{\mathbf{2}}]_{ij} = \Sigma_k a_{ik} a_{kj}$$

$$[\mathbf{A}^{\mathbf{3}}]_{ij} = \Sigma_k \Sigma_h a_{ik} a_{kh} a_{hj}$$

Impact of a unit increase in demand

1-step (direct) path $\rightarrow \mathbf{A}$

2-step paths $\rightarrow \mathbf{A} \times \mathbf{A} = \mathbf{A}^2$

3-step paths $\rightarrow \mathbf{A} \times \mathbf{A}^2 = \mathbf{A}^3$

4-step paths $\rightarrow \mathbf{A} \times \mathbf{A}^3 = \mathbf{A}^4$

: : :

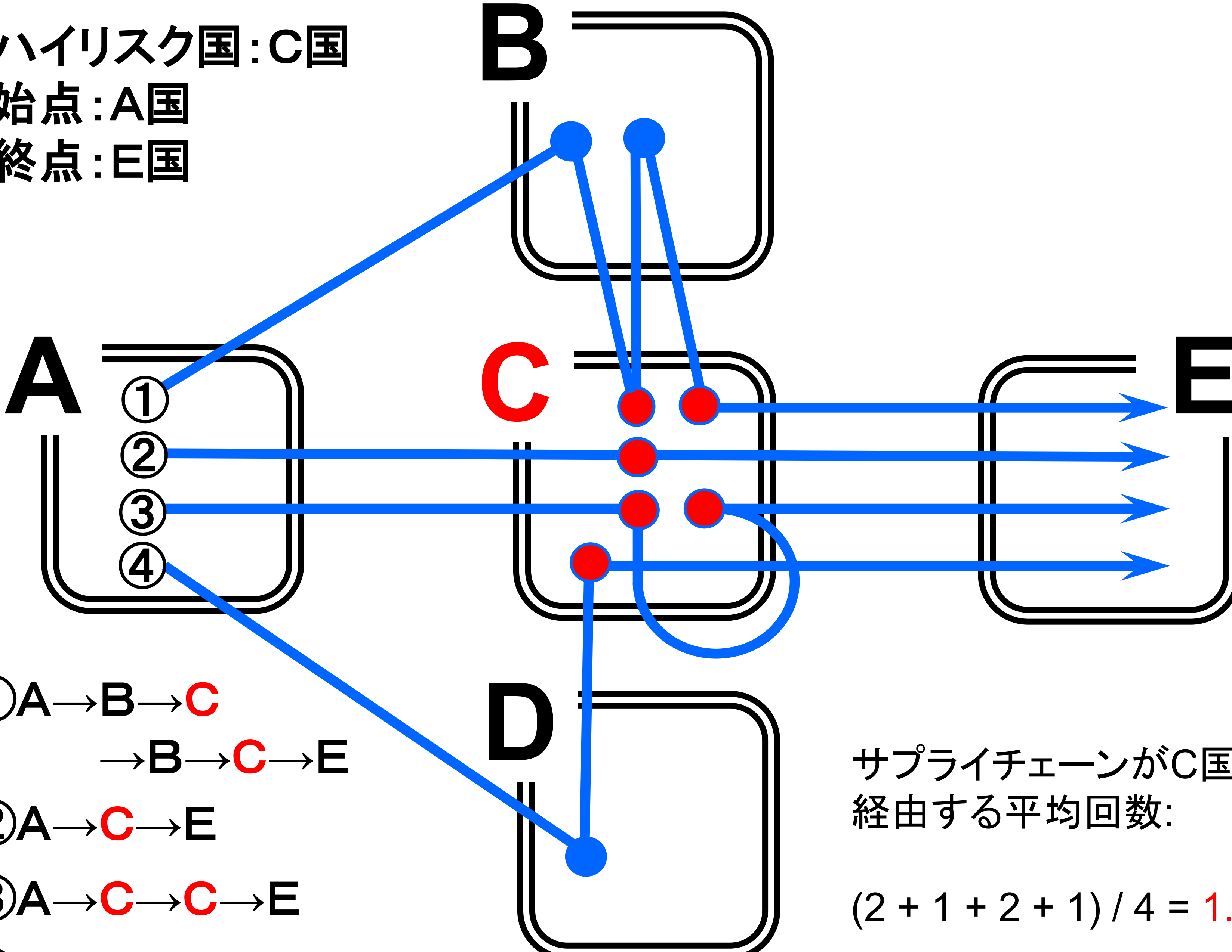
k-step paths $\rightarrow \mathbf{A} \times \mathbf{A}^{k-1} = \mathbf{A}^k$

通過頻度指標
Pass-through frequency (PTF)

ある製品のサプライチェーン上に、
特定国の産業部門が
どのくらいの頻度で登場するか。

→ 特定国に対する地理的集中度

ハイリスク国：C国
始点：A国
終点：E国



①A→B→C
→B→C→E
②A→C→E
③A→C→C→E
④A→D→C→E

サプライチェーンがC国を
経由する平均回数:

(2 + 1 + 2 + 1) / 4 = 1.5

Pass-through Frequency (PTF)

$$
\begin{aligned}
&{f_{(t)}}_{s_1 s_k} \\
&= c_{(t)} \cdot \frac{a_{s_1 s_k}}{[\boldsymbol{L} - \boldsymbol{I}]_{s_1 s_k}} \\
&\quad + \sum_{k=3}^{\infty} \sum_{s_{2,\ldots,} s_{k-1}} \left(c_{(t)} \cdot \frac{a_{s_1 s_2}\, a_{s_2 s_3}\, a_{s_3 s_4} \cdots a_{s_{k-1} s_k}}{[\boldsymbol{L} - \boldsymbol{I}]_{s_1 s_k}} \right)
\end{aligned}
$$

Liang et al. (2016)
Structural Betweenness Centrality

$c_{(t)}$: the number times that sector *t* of a high-risk country appears in the path between sector s_1 and sector s_k

Pass-through Frequency (PTF)

$$
\begin{aligned}
& f_{(t)_{s_1 s_k}} \\
& = c_{(t)} \cdot \frac{\underbrace{a_{s_1 s_k}}_{\text{Direct propagation path}}}{[\boldsymbol{L}-\boldsymbol{I}]_{s_1 s_k}} \\
& \quad + \sum_{k=3}^{\infty} \sum_{s_{2,\ldots,} s_{k-1}} \left(c_{(t)} \cdot \frac{\overbrace{a_{s_1 s_2} a_{s_2 s_3} a_{s_3 s_4} \cdots a_{s_{k-1} s_k}}^{\text{Indirect propagation path}}}{[\boldsymbol{L}-\boldsymbol{I}]_{s_1 s_k}} \right)
\end{aligned}
$$

Impact propagation of a particular production path

$c_{(t)}$: the number times that sector *t* of a high-risk country appears in the path between sector s_1 and sector s_k

$$a_{gh}\, a_{hi}\, a_{ij}$$

The amount of *g* sector products required to produce one unit output of sector *j* via production in sector *i* and sector *h*.

Input coefficient sequence as a **propagation path**

Pass-through Frequency (PTF)

Direct propagation path

Impact propagation of a particular production path

Indirect propagation path

$$f_{(t)_{s_1 s_k}} = c_{(t)} \cdot \frac{a_{s_1 s_k}}{[\boldsymbol{L} - \boldsymbol{I}]_{s_1 s_k}} + \sum_{k=3}^{\infty} \sum_{s_{2,\ldots,} s_{k-1}} \left(c_{(t)} \cdot \frac{a_{s_1 s_2} \, a_{s_2 s_3} \, a_{s_3 s_4} \ldots \, a_{s_{k-1} s_k}}{[\boldsymbol{L} - \boldsymbol{I}]_{s_1 s_k}} \right)$$

Total impact propagation of all paths between s_1 and s_k

$c_{(t)}$: the number times that sector *t* of a high-risk country appears in the path between sector s_1 and sector s_k

Pass-through Frequency (PTF)

$$f_{(t)s_1 s_k} = c_{(t)} \cdot \boxed{\frac{a_{s_1 s_k}}{[\boldsymbol{L}-\boldsymbol{I}]_{s_1 s_k}}} + \sum_{k=3}^{\infty} \sum_{s_{2,\ldots,}s_{k-1}} \left(c_{(t)} \cdot \boxed{\frac{a_{s_1 s_2} a_{s_2 s_3} a_{s_3 s_4} \cdots a_{s_{k-1} s_k}}{[\boldsymbol{L}-\boldsymbol{I}]_{s_1 s_k}}} \right)$$

An impact share of
a particular production path

The weighted average of the number of *t* sector emergence along production paths

→ The *frequency* that a supply chain passes through the industrial sectors of a high-risk country

Pass-through Frequency (PTF)

$$
{f_{(t)}}_{s_1 s_k}
$$

$$
= c_{(t)} \cdot \frac{a_{s_1 s_k}}{[\boldsymbol{L} - \boldsymbol{I}]_{s_1 s_k}}
$$

$$
+ \sum_{k=3}^{\infty} \sum_{s_{2,\dots,} s_{k-1}} \left(c_{(t)} \cdot \frac{a_{s_1 s_2}\, a_{s_2 s_3}\, a_{s_3 s_4} \cdots a_{s_{k-1} s_k}}{[\boldsymbol{L} - \boldsymbol{I}]_{s_1 s_k}} \right)
$$

$$
\Leftrightarrow \quad {f_{(t)}}_{ij} = \frac{\left[\boldsymbol{L}\boldsymbol{J}_{(t)}\boldsymbol{L} - \boldsymbol{J}_{(t)}\right]_{ij}}{[\boldsymbol{L} - \boldsymbol{I}]_{ij}}
$$

$\boldsymbol{J}_{(t)}$: a matrix with 1 for (t, t)th element and zeros elsewhere.