Automation, Human Capital and Welfare: The Stochastic Uzawa-Lucas Approach☆

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Abstract

Human capital, in the form of embodied knowledge or skills, is under threat of automation. To examine the effect of it on a macroeconomy, I develop the Uzawa-Lucas model in which the accumulation of human capital follows the stochastic process. With two equality constraints, the closed-form solution gets obtainable. It shows that increase in uncertainty about the human capital accumulation reduces the welfare of households and causes human capital contraction. Moreover, optimal consumption appears to depend on efficiency of human capital, not its stock.

Keywords: Human Capital, Stochastic Growth, Closed-Form Solution, Automation
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1. Introduction

In the age of automation and artificial intelligence (AI), rapid technological progress is threatening human capital in the form of embodied knowledge or skills. In order to examine the effect of automation on existing human capital and our macroeconomy, I develop the Uzawa-Lucas model in which the human capital accumulation follows the stochastic process. Imposing two equality constraints, I obtain the closed-form solution. It shows that increase in uncertainty about the accumulation of human capital reduces the household welfare and leads to human capital contraction. Furthermore, consumption is shown to depend upon efficiency of human capital, not its stock.

Automation is taking place globally at unprecedented speed. Figure A.1 plots the average robot sales over the last 15 years. Prior to the financial crisis of 2008, they had been roughly flat. However, after the crisis, the *average* annual supply rose sharply to about 183,000 units. We can clearly see that the demand for industrial robots has accelerated considerably since 2009. Moreover, the Hodrick-Prescott filtered trend indicates that it is likely to continue

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on a trend upwards. Such recent and rapid automation raises the obvious question: What impact will automation have on our macroeconomy as a whole?

Two views, the pessimistic and optimistic, are taken in the literature. The pessimistic view is that labors are increasingly being replaced by computers and thus unemployed, as Brynjolfsson and McAfee (2012[8]) argue. Today, machines can perform a variety of routine tasks and displace labors from tasks they were previously performing. Taking concrete examples, Frey and Osborne (2017[16]) estimate the probability of computerization for 702 detailed occupations. They conclude that 47 percent of US labors had jobs at high risk of potential automation over the next two decades[1]. Acemoglu and Restrepo (2017[2]) document that one more robot per thousand workers reduces the employment to population ratio by about 0.18 - 0.34 percentage points and aggregate wages by 0.25 - 0.5 percent. Ultimately, automation and AI revolution may lead to a substantially smaller role for labor in the workplace of the future (Acemoglu, Autor, Dorn, Hanson and Price (2014[1]))[2].

The optimistic view is that, although technology replaces some jobs, it creates new jobs[3], or some jobs cannot be automated. On the former point, the Deloitte (2015[15]) report documents that "while technology has potentially contributed to the loss of approximately 800,000 lower-skilled jobs, there is equally strong evidence to suggest that it has helped to create nearly 3.5 million new higher-skilled ones in their place." Put differently, our macroeconomy *as a whole* benefits from automation, from viewpoint of job creation.

On the latter point, Kaku (2012[24]) argues that jobs requiring pattern recognition and common sense cannot be automated. Taking two of his examples, lawyers can survive, because the ultimate interpretation of the law boils down to a value judgment, where computers are deficient. People in the arts too can survive, because creating art that inspires, intrigues, evokes emotions, and thrills us, all involves common sense and is thus beyond the capability of computers. Arntz, Gregory and Zierahn (2016[3]) take worker heterogeneity into account and find that, across the 21 OECD countries, only 9 percent of jobs are at risk of automation.

The truth of automation effects on our macroeconomy may lie between these two extremes, but which effect is dominant *today*? Figure A.2 shows the unemployment rate across G7 countries over the last three decades. We can see that, after the global crisis, unemployment rates show the downward trend among G7 economies, probably except for Italy. Thus, in terms of unemployment rates, the optimistic view seems dominant.

However, note that US unemployment rates circa 2000 - the period of information technology (IT) revolution or the rise of "New Economy," together with the dot-com bubble - behave interestingly. Between 2000 and 2003, as the pessimistic view suggests, we can see an *upward* trend in the US unemployment rate, in spite of the productivity growth. Certainly,

[1] One of job groups with strong probability of being computerized is sales and services, such as cashiers and telemarketers.

[2] The early theoretical contribution on this line of research is Zeira (1998[42]). On the structural change, Jones (2016[23]) gives an example of the agricultural share of the US labor force. It went from two-third to only 2 percent, largely because of mechanization and technological change.

[3] It is also possible for technologies to complement and increase the productivity of certain types of skills.

part of its increase may have been due to the burst of the dot-com bubble. Nonetheless, it may well be the case that it was partly because of automation.

As an alternative, Figure A.3 displays measures of human capital stock among G7 countries for people aged 15 to 64, over the past 145 years (in 5-year intervals) , recently constructed by Lee and Lee (2016[27]). It shows that G7 countries all have steadily accumulated human capital. Going over, focusing on the US, we can find the interesting patterns. In the US history, there has been human capital *contraction* twice (indicated by the shaded areas).

The first would be triggered by World War II. It is crystal clear that it destroyed human capital of nations. What is less clear and more interesting is the destruction of human capital for years between 2000 and 2005. Why did human capital contract during this period? The first possible explanation is that human capital investment became weak, because of, again, the burst of dot-com bubble and subsequent economic downturn. But the second possibility is that human capital, in the form of embodied knowledge or skills, became obsolete owing to automation, induced by the IT revolution circa 2000. For instance, although retail cashiers have accumulated knowledge to do their job, they have been replaced by self-checkout machines.

Given the upward trend over the very long run, the pessimistic view seems valid, in terms of human capital accumulation. Incidentally, Comin and Hobijn (2010[11]) document that technology adoption lags (i.e. the number of years between the date a technology was invented and the date it was adopted in the country) have been shrinking over time. Their estimate suggests that technologies invented 10 years later are, on average, adopted 4.3 years faster. This implies that newly invented robots will increasingly be in use at faster pace in upcoming several decades. Correspondingly, though human capital data since 2010 is not available, it is reasonable to expect that human capital contraction will take place again in the *future*, due to rapid automation visible in Figure A.1.

Studies cited above, being either pessimistic or optimistic, focus on the effect of automation mainly on the labor market, such as employment and wage inequality. The purpose of this paper is neither to justify which view is correct, nor to examine automation effects on labor market outcomes. In sharp contrast, I examine the effect of automation on macroeconomy as a whole, not only on the labor market. Specifically, I analyze how the risk of automation affects the accumulation of human capital and household welfare. Although it seems quite natural to presume that people are worse off due to technological unemployment, it must be the case that they are rather *better off*, if technology complements and increases the productivity of their skills.

For example, bank tellers may be better off thanks to automated teller machines (ATMs), for they no longer have to do routine tasks which ATMs instead do today. In other words, existing literature cannot evaluate how automation affects household welfare, or at best, can claim that automation is welfare-reducing *indirectly* by invoking the possibility of technological unemployment. Here, I show that increased automation risk *directly* reduces household welfare. To this end, I use the stochastic Uzawa-Lucas model to investigate an economy where human capital of labors are always and instantaneously at risk of being obsolete, because of rapid automation.

This paper is organized as follows. Section 2 briefly reviews recent developments of the

Uzawa-Lucas model. Section 3 sets up the stochastic version of the Uzawa-Lucas model and discusses its implications. Concluding remarks appear in Section 4.

2. Literature Review

In what follows, I will use the streamlined Uzawa (1965[39])-Lucas (1988[28]) model in which the accumulation of human capital follows the stochastic process. This model is suitable for my purpose, because human capital is an explicit input. In this section, I briefly review current developments of this model.

Some authors study the standard Uzawa-Lucas model in a variety of dimensions. Despite its simplicity, this model can be used to analyze various economic phenomena. Also, finding the closed-form solution to this model, especially in the context of stochastic environment, is a hot topic in the literature[4].

Bethmann (2007[6]) studies the discrete-time stochastic Uzawa-Lucas model in which technology follows the first-order autoregressive, or AR(1), process. Hiraguchi (2009[18]) finds the closed-form solution to the deterministic continuous-time Uzawa-Lucas model in the presence of externalities, with no restrictions on parameters. Bethmann and Reiß (2012[7]) use the deterministic Uzawa-Lucas model in continuous time, to illustrate the simple way of computing the value function. Gorostiaga, Hromocová and López-García (2013[17]) examine taxation in the discrete-time Uzawa-Lucas model under certainty. Naz, Chaudhry and Mahomed (2016[33]) derive the set of explicit solutions to the deterministc Uzawa-Lucas model in continuous time, by employing the partial Hamiltonian approach. The Uzawa-Lucas model has an exogenous parameter that indicates how efficient human capital accumulation is. Analyzing the continuous-time Uzawa-Lucas model under certainty, Kuwahara (2017[26]) first endogenizes that parameter.

None of the above inquires into the stochastic Uzawa-Lucas model in continuous time. Recently, Bucci, Colapinto, Forster and La Torre (2011[9]) and Hiraguchi (2013[19]) present the continuous-time stochastic Uzawa-Lucas model in which technology follows a geometric Brownian motion[5]. Both papers find the closed-form solutions despite their relatively general functional forms of production function. Since their models are, in a way, akin to mine, I spell out some of their contributions.

Bucci et al.[9] first find the closed-form solution in the stochastic Uzawa-Lucas model in continuous time, in spite of its generality (for instance, the power utility function and the

[4]In fact, generally speaking, the explicit solution rarely exists in continuous time under uncertainty (see Turnovsky (2000) and Wälde (2011[40])). Although Chang (2004, Chapter 5) collects some class of models in which the closed-form solution exists, there is yet no algorithm to find it. Even found, it usually sets off unpleasant features, such as negative consumption or negative capital-output ratio. Obtaining the closed-form solution which does not suffer from those unpleasant characteristics has been, and will be, a formidable task.

[5]The continuous-time approach has advantages over the discrete-time approach. One of them is that, thanks to Itô's Lemma, the Hamilton-Jacobi-Bellman equation is not stochastic. This is not true, in general, in discrete time. The benefit of continuous-time modeling is extensively surveyed by Brunnermeier and Sannikov (2016[4]).

generalized Cobb-Douglas production function á la Mankiw, Romer and Weil (1992[30])). They impose *two* equality constraints on parameters to obtain the explicit solution. It shows that "technology shocks" reduce the optimal level of consumption and time devoted to the production of goods.

Hiraguchi[19] revisits Bucci et al.[9] and finds that, without changing their model at all, the closed-form solution can obtain with *one* parameter restriction only. Moreover, Hiraguchi[19] proves that the value function of Buccie et al.[9], de facto, does not safisfy the Hamilton-Jacobi-Bellman (HJB) equation, unless the product of the initial value of human capital and technology is constant, and figures out the new correct value function. It shows that "technology shocks" have nothing to do with the optimal level of consumption and increase time devoted to the goods production, in sharp contrast to Bucci et al.[9].

Their breakthrough makes it clear how technology shock affects aggregate macroeconomic variables. All the same, they are silent on several issues, because they are primarily concerned with the mathematical aspect of value function. First, what is meant by the "technology shock"? Since the shock is exogenous and they simply use the term "technology shock," interpreting its effect on our economy in plain English, is extremely hard. Although I also use the exogenous stochastic process, my "shock" has a clear economic meaning.

Second, they do not ask important questions that *only* stochastic models can ask: "How does increasing uncertainty affect household welfare?" In their context, what impact does "technology shock" have on the welfare of households? They do not answer this question, and do not even ask. The advantage of stochastic modeling is that we can assess the variables of interest, both under certainty and uncertainty. This is the motivation behind my inquiry into the relationship between automation risk and household welfare.

3. The Model

Throughout, I make three assumptions: (i) The total number of workers L equals unity, so that the per capita terms are equivalent to the aggregate terms. (ii) I abstract from the technological progress A. (iii) I abstract from depreciation δ of both human and physical capital.

The first assumption greatly simplifies the analysis below. The second and third assumptions enable me to obtain the closed-form solution. I admit the importance of the technological progress and depreciation of capital. However, the point of this paper is to examine the effect of the stochastic accumulation of human capital in the most transparent model, not to be as general as it could be, like Bucci et al.[9].

3.1. Capital Accumulation

A representative household is endowed with one unit of time and uses all of that. It either works or learns. There is no other use of time. Let $u \in (0, 1)$ denote the fraction of time spent working to produce final goods $Y(t)$. Correspondingly, $1 - u$ represents the fraction of time spent learning. The amount of leisure is fixed exogenously, so there is no choice about it.

The accumulation of human capital $H(t)$ is stochastically governed by the following rule:

$$dH(t) = b(t)(1-u(t))H(t)dt + \sigma H(t)dz_t \tag{1}$$

where dz_t is the increment of the Wiener process such that $E(dz_t) = 0$ and $Var(dz_t) = dt$. $\sigma > 0$ is the diffusion coefficient (*in principle*, if $\sigma = 0$, then we recover the deterministic limit). It can be interpreted as the degree of automation risk. Higher σ means that labors are more easily replaced by computers, and vice versa. $b > 0$ is a exogenous parameter that indicates how efficient human capital accumulation is. The initial stock of human capital $H(0) = H_0 > 0$ is given.

The economy-wide resource constraint is

$$dK(t) = \underbrace{[u(t)H(t)]^{\alpha}K(t)^{1-\alpha}}_{\equiv Y(t)} dt - C(t)dt \tag{2}$$

where $K(t)$ is physical capital. $\alpha \in (0,1)$ represents the labor share of income in the Cobb-Douglas production function. $C(t)$ denotes consumption of the final good. The initial stock of physical capital $K(0) = K_0 > 0$ is given as well.

3.2. Household

Preferences of a representative household are given by the standard constant relative risk aversion (CRRA) utility function:

$$E\int_0^{\infty} e^{-\rho t}\frac{C(t)^{1-\phi}-1}{1-\phi}dt \tag{3}$$

where E is the mathematical expectation operator with respect to the information set available to the representative household. $\rho > 0$ is the subjective discount rate of that. $\phi > 0$ is the coefficient of relative risk aversion (or the inverse of the elasticity of intertemporal substitution). The representative household maximize its utility (3) subject to (1) and (2) under uncertainty.

3.3. Optimization

In order to solve the optimization problem, let $J(H,K)$ be the value function. Then, the associated HJB equation reads

$$\max_{\{C_t,u_t\}}\left(\frac{C(t)^{1-\phi}-1}{1-\phi} - \rho J(K,H) + J_K\frac{dK}{dt} + J_H\frac{dH}{dt} + \frac{1}{2}J_{HH}\frac{(dH)^2}{dt}\right) \tag{4}$$

i.e.

$$\max_{\{C_t,u_t\}}\left(\frac{C(t)^{1-\phi}-1}{1-\phi} - \rho J(K,H) + J_K([uH]^{\alpha}K^{1-\alpha} - C) + J_H b(1-u)H + \frac{1}{2}J_{HH}\sigma^2H^2\right) \tag{5}$$

where $J_K \equiv \frac{\partial J}{\partial K}$, $J_H \equiv \frac{\partial J}{\partial H}$ and $J_{HH} \equiv \frac{\partial^2 J}{\partial H^2}$. First-order conditions result in

$$C = J_K^{-\frac{1}{\phi}} \tag{6}$$

$$u = \frac{K}{H}\left(\frac{\alpha J_K}{bJ_H}\right)^{\frac{1}{1-\alpha}} \tag{7}$$

Substituting these first-order conditions back to the HJB equation (5) and rearranging, we get

$$0 = \frac{\phi}{1-\phi}J_K^{\frac{\phi-1}{\phi}} - \frac{1}{1-\phi} - \rho J(K,H) + J_K K\left(\frac{\alpha J_K}{bJ_H}\right)^{\frac{\alpha}{1-\alpha}} + J_H\left(bH - bK\left[\frac{\alpha J_K}{bJ_H}\right]^{\frac{1}{1-\alpha}}\right) + \frac{\sigma^2}{2}J_{HH}H^2 \tag{8}$$

With this equation, our task now is to "guess and verify" the functional form of the value function and find the closed-form solution, if it exists. Unfortunately, with no restraint, the closed-form solution is latent. Howbeit, with two equality constraints, we *can* obtain the explicit analytical solution. It can be summarized as follows:

Proposition. *If we impose the following parameter constraints,*

$$\phi = \alpha = \frac{1}{2} \tag{9}$$

$$b = 2\rho \tag{10}$$

then there exists the closed-form solution (that satisfies the transversality condition or TVC) of the form

$$J(K,H) = \Upsilon_\sigma \Lambda_\rho \sqrt{H} + \sqrt{\Lambda_\rho}\sqrt{K} - \Lambda_\rho \tag{11}$$

where

$$\Upsilon_\sigma \equiv \frac{1}{2\sigma} \tag{12}$$

$$\Lambda_\rho \equiv \frac{2}{\rho} \tag{13}$$

Together with first order conditions (6) and (7), the value function (11) yields the expressions for control variables:

$$C = bK \tag{14}$$

$$u = \frac{\sigma^2}{16b} \tag{15}$$

Besides, we can be sure that $u \in (0,1)$, so long as

$$\sigma \in (0, 4\sqrt{b}) \tag{16}$$

Proof. See Appendix A[6].

3.4. Discussion

I first discuss the robustness of two equality constraints (9) and (10) and in turn comment on the rest of equations in the Proposition.

3.4.1. Robustness

Equation (9) says that the coefficient of relative risk aversion ϕ and the labor share of income α both equal one-half. This equality constraint is the special case of Xie (1994[41]), Rebelo and Xie (1999[34]) and Smith (2007[37]). They all impose $\phi = \alpha$ to obtain the explicit analytical solution[7]. In the present economy, I am assuming that the labor share of income equals the physical capital share. You may object that the labor share should be two-third[8]. Be that as it may, recent evidence (especially Karabarbounis and Neiman (2014[25])) shows that there has been a remarkable decline in the labor share across countries (and correspondingly the rising capital share) since 2000 or so[9]. Accordingly, the first equality constraint (9) seems solid.

The second equality (10) merely says that the learning efficiency parameter b is twice as large as the subjective discount rate ρ. How restrictive is this assumption? When simulating the Uzawa-Lucas model, Barro and Sara-i-Martin (2004[4], page 260) use $b = 0.11$ (fixing $\alpha = 0.5$, as I do here). To this extent, (10) would be cogent if ρ is close to 0.05. As a matter of fact, $\rho = 0.05$ is widely used in the literature (for instance, see Moll (2014[32]) and Caballé and Santos (1993[12])). In addition, Bucci et al[9] use $b = 0.09$ and $\rho = 0.04$, implying that b is about twice as large as ρ. In this manner, the second equality constraint (10) appears to be persuasive as well.

3.4.2. Macroeconomic Implications

Equation (11) is the unique value function that satisfies the HJB equation. It is inversely related to the degree of automation risk σ. To get the sense of the value function, see Figure A.4[10]. The left panel is the case for $\sigma = 0.005$ and thus virtually deterministic. The right

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[6]You may realize that, as σ goes to zero (i.e. as we recover the deterministic limit), the constant term (12) cannot be defined, while u (15) can be zero. This possibility is at odds. Nonetheless, it is safely ruled out by the inequality (16). If we set $\sigma = 0$, this inequality would be violated and u would no longer lie in the unit interval.

[7]The assumption of $\phi = 0.5$ is under debate. Smith[37] argues that $\phi < 1$, whereas Hiraguchi[18] says that $\phi \approx 5.0$.

[8]See, for example, Bernanke and Gürkaynak[5] or Romer (2012[35], page 25).

[9]The global decline of labor share is extensively discussed in International Monetary Fund (2017b[22], Chapter 3). At the other extreme, Karabarbounis and Neiman[25] report that the *labor* share is approaching one-third in China.

[10]I use MATLAB R2016b (Version 9.1, MATLAB and Simulink Student Suite) to create the following three Figures. Matlab code is available upon request.

panel is the case for $\sigma = 0.1$ and allows for some degree of uncertainty. Figure A.4 clearly shows that increase in σ reduces the welfare of the representative household (measured by the value function). People in this economy are risk-averse (remember the CRRA utility), so they prefer the deterministic environment to the stochastic one. As regards the policy, it should aim to reduce uncertainty about the human capital accumulation, in order to improve the household welfare. The intuitive explanations follows below.

Equation (14) shows that optimal consumption is the constant fraction of the physical capital stock, or put differently, consumption-physical capital ratio is equal to learning efficiency b. The relationship is visualized in Figure A.5. You can see two lines, blue one ($b = 0.11$, used by Barro and Sala-i-Martin[4]) and red one ($b = 0.09$, used by Bucci et al.[9]). The Figure demonstrates that, given K, higher b leads to stronger C.

That consumption can strengthen by enhancing learning efficiency is the solution to the puzzle of Hiraguchi[19]. Hiraguchi ([19], page 137) wonders that "We cannot not find the intuitive explanation why the current consumption level c is independent of the TFP level A and the human capital level H." The Figure makes clear: It is not the *stock* of human capital that matters for consumption. It is *efficiency* of human capital that does matter for consumption.

Put differently, creating millions of robots itself does not matter, if not at all. Rather, improving their quality is the key to stimulating consumption. If we can regard the efficiency parameter b as quality of human capital accumulation, it would be consistent with empirical findings that stress the importance of quality of human capital, such as Schoellman (2012[36]) and Manuelli and Seshadri (2014[31]).

Equation (15) says that u is a decreasing function of b, but an increasing function of σ. The latter is consistent with Hiraguchi[19], though his σ represents the TFP shock, not automation risk. The first point is intuitive: if the accumulation of human capital becomes more efficient (i.e. b increases), people are more inclined to allocate their time to learning or acquiring new skills (i.e. u decreases), resulting in the higher growth rate of human capital.

The second point is more interesting and deserves of spelling out. Why do people spend more time on production when investment in human capital becomes riskier? In order to understand this point, see Figure A.6[11]. It illustrates, in general terms, how increasing σ affects the optimal choice of u given b. For any b, we can see that increase in σ raises u.

One interpretation is that, when σ is higher, your effort to acquire new skills may result in waste of time, because newly invented machines may already equipped with skills to do that job. Finding skill acquisition unattractive, people seek for jobs which they can do with their current skills. As a consequence, people prefer spending more time on production to learning or skill acquisition under uncertainty.

This has a marked policy implication. Above, we saw that reducing uncertainty improves the welfare of households and enhances the growth rate of human capital. To achieve this, at the same time, the economy has to bear instantaneous output contraction, because of reductive time spent working u. In this sense, the policy faces the trade-off: the welfare-improving

[11]Strictly speaking, I use $\sigma = 0.0001$ for the case of $\sigma = 0$ displayed in the Figure, to ensure that the inequality (16) holds.

policy accompanied by positive human capital growth gives rise to output contraction. By the same token, the welfare-reducing policy accomplishes the opposite. This conclusion can only be pinned down by the theory of stochastic growth. The deterministic growth theory can never dig out this finding. What is more, it is the consequence of modeling the stochastic accumulation of human capital. It is even absent in Bucci et al.[9]'s and Hiraguchi[19]'s modeling of stochastic technology following a geometric Brownian motion.

The inequality (16) ensures that u is in the unit interval. Though directly limiting the size of uncertainty may seem grim, it is simpler. For example, to ensure $u \in (0, 1)$, Hiraguchi[19] has to impose four inequality constraints for u. Here, however, with only one inequality constraint, u lies safely in the unit interval. Furthermore, this limit seems not to be radically restrictive. See again Figure A.6.

It illustrates the tolerance level of σ as we vary b. We can see that, given b, increase in σ makes it harder to ensure $u \in (0, 1)$. But when $b = 0.1$, the approximate value used by Barro and Sara-i-Martin[4], even if we increase σ from the low of 0% to 40%, to 80%, and to the high of 120%, u still lies in the unit interval. Truly, it is desirable if we can ensure $u \in (0, 1)$ with no restriction on σ. Yet as Figure A.6 makes clear, σ can be relatively large. Therefore, the analysis of this paper is not limited to the neighborhood of deterministic paths, i.e. $\sigma = 0$.

4. Concluding Remarks

This paper analyzes the effect of automation on the human capital, in the form of embodied knowledge or skills, and on our macroeconomy. To that end, I develop the Uzawa-Lucas model in which human capital follows the stochastic process. Two equality constraints allow for the closed-form solution, even in continuous time under uncertainty. It shows that the increasing risk of automation reduces household welfare, and leads to human capital contraction. Moreover, consumption is shown to hinge upon efficiency of human capital, not its stock. These results cannot obtain in the deterministic model or the model with stochastic technology following a geometric Brownian motion.

The continuous-time model under uncertainty, in most cases, does not admit the closed-form solution. It is instead possible to resort to, for instance, the value function iteration and then get the approximate policy function that links the optimal level of control variables and state variables (here, consumption and the physical capital stock). However, giving up the analytical solution gives rise to the "black box." We may be able to understand the major properties of the model, but the solution procedure is likely to leave the important mechanism blind. Taking an example, the relationship between consumption and efficiency of human capital accumulation would not be identified, if I resorted to the value function iteration. We should welcome the increasing computing power in economics, but the heart of economics is really found "by hand."

I end with two possible directions for the future research. The first is obvious. It would be desirable if the analytical solution obtains without (in)equality constraint(s) or without abstracting from depreciation of capital. The second is more important but formidable: to study the stochastic model in which both the human capital accumulation and technology follow the stochastic process. Such a modeling will unveil the important mechanism of the interaction between human capital and TFP, recently emphasized by Madsen (2014[29]) and Cinnirella and Streb (2017[14]). Maybe we can start with the case in which two stochastic processes are uncorrelated, and then proceed to the correlated case. Whether we can find the closed-form solution for such a class of models is uncertain at this stage. But the possibility is here to stay.

Appendix A. Value Function and TVC

In this Appendix, I show you how to find the functional form of the value function. In addition, I prove that the proposed value function satisfies the TVC, hence unique, because of the verification theorem (see Chang (2004[13], Chapter 4) for details of this theorem).

Appendix A.1. Value Function

I use the standard "guess and verify" method to find the closed-form solution. The exposition here is based on the Appendix A of Bucci et al.[9]. What I present is, in effect, similar to the method of undetermined coefficients.

I postulate the tentative value function of the form:

$$J(H,K) = T_H H^{\theta_1} + T_K K^{\theta_2} + X \tag{A.1}$$

where T_H, T_K, X, θ_1 and θ_2 are all unknown constants to be determined. The resulting first and second partials with respect to physical and human capital are:

$$J_K = \theta_2 T_K K^{\theta_2 - 1} \tag{A.2}$$

$$J_H = \theta_1 T_H H^{\theta_1 - 1} \tag{A.3}$$

$$J_{HH} = \theta_1(\theta_1 - 1) T_H H^{\theta_1 - 2} \tag{A.4}$$

Substituting equations (A.1), (A.2), (A.3), (A.4) and first-order conditions (6), (7) into the HJB equation (5), we get

$$0 = \frac{K^{1-\theta_2}}{\theta_2 T_K} - \rho T_K K^{\theta_2} + \frac{(\theta_2 T_K)^2}{4b\theta_1 T_H H^{\theta_1 - 1}} K^{2\theta_2 - 1} - \rho X - 2 + \theta_1 T_H b H^{\theta_1} - \rho T_H H^{\theta_1} + \frac{\sigma^2}{2}\theta_1(\theta_1 - 1) T_H H^{\theta_1} \tag{A.5}$$

Setting $\theta_1 = \theta_2 = \frac{1}{2}$ yields

$$0 = \frac{2}{T_K}\sqrt{K} - \rho T_K \sqrt{K} + \frac{{T_K}^2}{8bT_H}\sqrt{H} - \rho X - 2 + \frac{1}{2} T_H b \sqrt{H} - \rho T_H \sqrt{H} - \frac{\sigma^2}{8} T_H \sqrt{H} \tag{A.6}$$

Collecting terms,

$$0 = \sqrt{K}\left(\frac{2}{T_K} - \rho T_K\right) + \sqrt{H}\left(\frac{{T_K}^2}{8bT_H} + \frac{b}{2}T_H - \rho T_H - \frac{\sigma^2}{8}T_H\right) - (\rho X + 2) \tag{A.7}$$

This equation must be satisfied for all K, H and X. Therefore, the expression in the first parenthesis should equal zero, giving

$$T_K = \sqrt{\frac{2}{\rho}} \tag{A.8}$$

Likewise, the expression in the second parenthesis yields (imposing the equality constraint (10), i.e. $b = 2\rho$, otherwise no explicit solution)

$$T_H = \frac{2}{b\sigma} \tag{A.9}$$

Finally, the third parenthesis implies

$$X = -\frac{2}{\rho} \tag{A.10}$$

Substituting (A.8), (A.9) and (A.10) back to the postulated value function (A.1) gives the value function in the main body, (11), whose constants are defined as in (12) and (13). The straightforward calculation proves the inequality (16).

Appendix A.2. TVC

The TVC to be satisfied is:

$$\lim_{t\to\infty} E[e^{-\rho t}\sqrt{K}] = \lim_{t\to\infty} E[e^{-\rho t}\sqrt{H}] = 0 \tag{A.11}$$

Because the inside of the square root must be positive (otherwise, physical and human capital stock can take the complex number), this TVC is obviously satisfied. More complicated and general proof of TVC can be found in the Appendix B of Hiraguchi[19].

References

[1] Acemoglu, Daron, David Autor, David Dorn, Gordon H. Hanson and Brendan Price. 2014. "Return of the Solow Paradox? IT, Productivity, and Employment in US Manufacturing." *American Economic Review: Papers & Proceedings* 104(5): 394 - 399.

[2] Acemoglu, Daron and Pascual Restrepo. 2017. "Robots and Jobs: Evidence from US Labor Markets." NBER Working Paper No. 23285 (March).

[3] Arntz, Melanie, Terry Gregory and Ulrich Zierahn. 2016. "The Risk of Automation for Jobs in OECD Countries: A Comparative Analysis." OECD Social, Employment and Migration Working Paper No. 189 (May).

[4] Barro, Robert J. and Xavier Sala-i-Martin. 2004. *Economic Growth* 2e. Cambridge, Massachusetts: MIT Press.

[5] Bernanke, Ben S. and Refet S. Gürkaynak. 2001. "Is Growth Exogenous? Taking Mankiw, Romer, and Weil Seriously." in Ben S. Bernanke and Kenneth Rogoff ed. *NBER Macroeconomics Annual* 16: 11 - 57.

[6] Bethmann, Dirk. 2007. "A Closed-Form Solution of the Uzawa-Lucas Model of Endogenous Growth." *Journal of Economics* 90(1): 87 - 107.

[7] Bethmann, Dirk and Markus Reiß. 2012. "Simplifying Numerical Analyses of Hamilton-Jacobi-Bellman Equations." *Journal of Economics* 107(2): 101 - 128.

[8] Brynjolfsson, Erik and Andrew McAfee. 2012. *Race Against The Machine: How the Digital Revolution is Accelerating Innovation, Driving Productivity, and Irreversibly Transforming Employment and the Economy.* Lexington, Massachusetts: Digital Frontier Press.

[9] Bucci, A., C. Colapinto, M. Forster and D. La Torre. 2011. "Stochastic Technology Shocks in an Extended Uzawa-Lucas Model: Closed-Form Solution and Long-Run Dynamics." *Journal of Economics* 103(1): 83 - 99.

[10] Brunnermeier, Markus. K. and Yuliy Sannikov. 2016. "Macro, Money, and Finance: A Continuous-Time Approach." in John Taylor and Harald Uhlig ed. *Handbook of Macroeconomics* Vol. 2B: 1497 - 1545.

[11] Comin, Diego and Bart Hobijn. 2010. "An Exploration of Technology Diffusion." *American Economic Review* 100(5): 2031 - 2059.

[12] Caballé, Jordi and Manuel S. Santos. 1993. "On Endogenous Growth with Physical and Human Capital." *Journal of Political Economy* 101(6): 1042 - 1067.

[13] Chang Fwu-Ranq. 2004. *Stochastic Optimization in Continuous Time.* New York: Cambridge University Press.

[14] Cinnirella, Francesco and Jochen Streb. 2017. "The Role of Human Capital and Innovation in Economic Development: Evidence from Post-Malthusian Prussia." *Journal of Economic Growth* 1 - 35.

[15] Deloitte. 2015. *From Brawn to Brains: The Impact of Technology on Jobs in the UK.* London: Deloitte LLP.

[16] Frey, Carl Benedikt and Michael A. Osborne. 2017. "The Future of Employment: How Susceptible Are Jobs to Computerisation?" *Technological Forecasting and Social Change* 114: 254 - 280.

[17] Gorostiaga, Arantza, Jana Hromcová and Miguel-Ángel López-García. 2013. "Optimal Taxation in the Uzawa-Lucas Model with Externality in Human Capital." *Journal of Economics* 108(2): 111 - 129.

[18] Hiraguchi, Ryoji. 2009. "A Note on the Closed-Form Solution to the Lucas-Uzawa Model with Externality." *Journal of Economic Dynamics & Control* 33: 1757 - 1760.

[19] Hiraguchi, Ryoji. 2013. "On A Closed-Form Solution to the Stochastic Lucas-Uzawa Model." *Journal of Economics* 108(2): 131 - 144.

[20] International Federation of Robotics. 2016. *World Robotics 2016.*

[21] International Monetary Fund. 2017a. *World Economic Outlook Database.*

[22] International Monetary Fund. 2017b. *World Economic Outlook: Gaining Momentum?* Washington, DC: International Monetary Fund.

[23] Jones, Charles I. 2016. "The Facts of Economic Growth." in John Taylor and Harald Uhlig ed. *Handbook of Macroeconomics* Vol. 2A: 3 - 69.

[24] Kaku, Michio. 2012. *Physics of the Future: The Inventions That Will Transform Our Lives.* New York: Penguin Books.

[25] Karabarbounis, Loukas and Brent Neiman. 2014. "The Global Decline of the Labor Share." *Quarterly Journal of Economics* 129(1): 61 - 103.

[26] Kuwahara, Shiro. 2017. "Multiple Steady States and Indeterminacy in the Uzawa-Lucas Model with Educational Externalities." *Journal of Economics*: 1 - 18.

[27] Lee, Jong-Wha and Hanol Lee. 2016. "Human Capital in the Long Run." *Journal of Development Economics* 122: 147 - 169.

[28] Lucas, Robert E. 1988. "On the Mechanics of Economic Development". *Journal of Monetary Economics* 22(1): 3 - 42.

[29] Madsen, Jakob B. 2014. "Human Capital and the World Technology Frontier." *Review of Economics & Statistics* 96(4): 676 - 692.

[30] Mankiw, N. Gregory, David Romer and David N. Weil. 1992. "A Contribution to the Empirics of Economic Growth." *Quarterly Journal of Economics* 107(2): 407 - 437.

[31] Manuelli, Rodolfo E. and Ananth Seshadri. 2014. "Human Capital and the Wealth of Nations." *American Economic Review* 104(9): 2736 - 2762.
[32] Moll, Benjamin. 2014. "Productivity Losses from Financial Frictions: Can Self-Financing Undo Capital Misallocation?" *American Economic Review* 104(10): 3186 - 3221.
[33] Naz, R., Azam Chaudhry and F.M. Mahomed. 2016. "Closed-Form Solutions for the Lucas-Uzawa Model of Economic Growth via the Partial Hamiltonian Approach." *Communications in Nonlinear Science and Numerical Simulation* 30: 299 - 306.
[34] Rebelo, Sergio and Danyang Xie. 1999. "On the Optimality of Interest Rate Smoothing." *Journal of Monetary Economics* 43(2): 263 - 282.
[35] Romer, David. 2012. *Advanced Macroeconomics* 4e. New York: McGraw-Hill.
[36] Schoellman, Todd. 2012. "Education Quality and Development Accounting." *Review of Economic Studies* 79(1): 388 - 417.
[37] Smith, William T. 2007. "Inspecting the Mechanism Exactly: A Closed-Form Solution to a Stochastic Growth Model." *B.E. Journal of Macroeconomics* 7(1): 1 - 31.
[38] Turnovsky, Stephen J. 2000. *Methods of Macroeconomic Dynamics* 2e. Cambridge, Massachusetts: MIT Press.
[39] Uzawa, Hirofumi. 1965. "Optimum Technical Change in An Aggregative Model of Economic Growth." *International Economic Review* 6(1): 18 - 31.
[40] Wälde, Klaus. 2011. "Production Technologies in Stochastic Continuous Time Models." *Journal of Economic Dynamics & Control* 35(4): 616 - 622.
[41] Xie, Danyang. 1994. "Divergence in Economic Performance: Transitional Dynamics with Multiple Equilibria." *Journal of Economic Theory* 63(1): 97 - 112.
[42] Zeira, Joseph. 1998. "Workers, Machines, and Economic Growth." *Quarterly Journal of Economics* 113(4): 1091 - 1117.

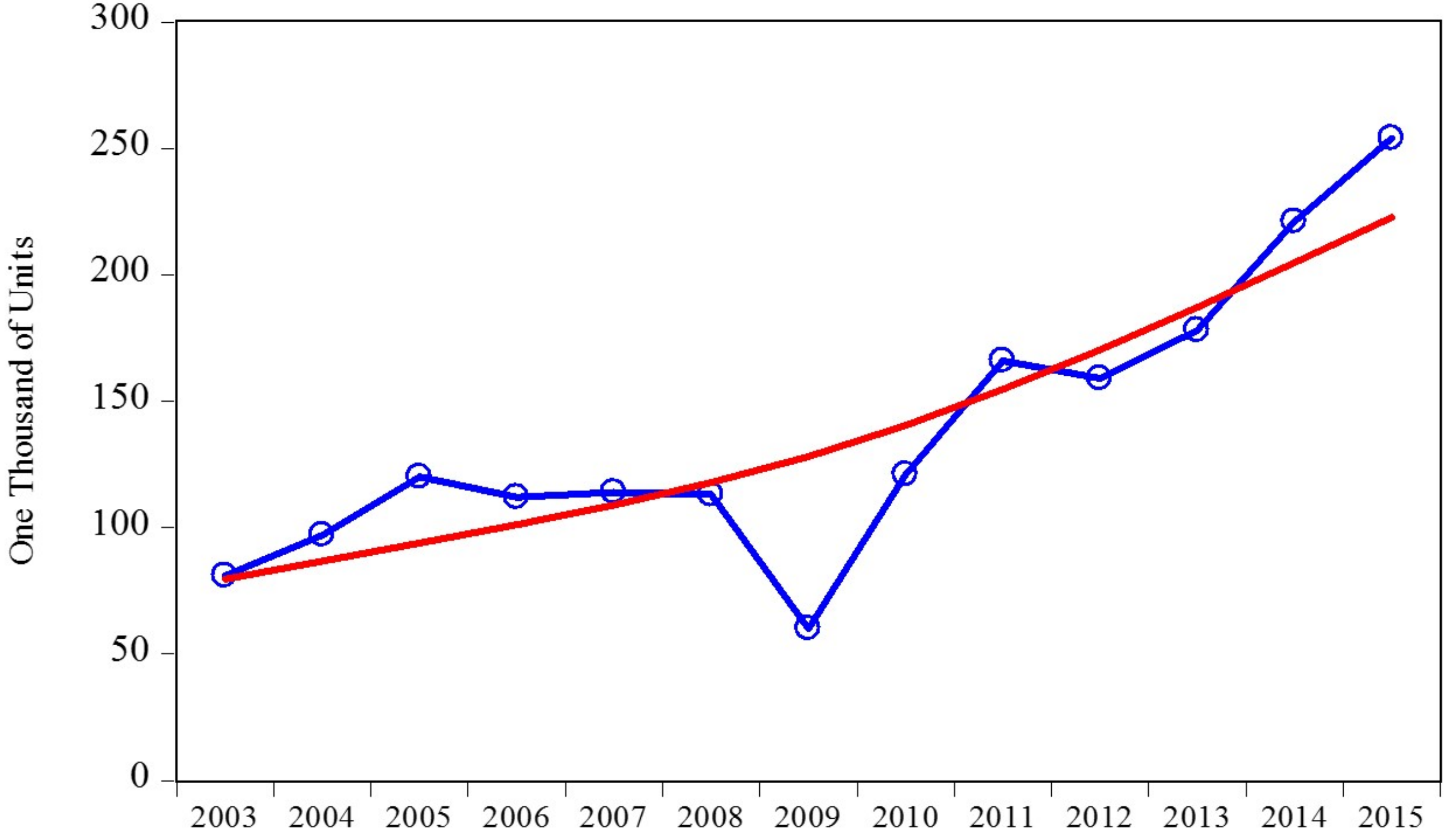

Figure A.1: Estimated worldwide annual supply of industrial robots. The average annual supply rose to about 183,000 units (an increase of about 59%) between 2010 and 2015. The red line shows the Hodrick-Prescott filtered trend. *Source*: Data from International Federation of Robotics (2016[20]).

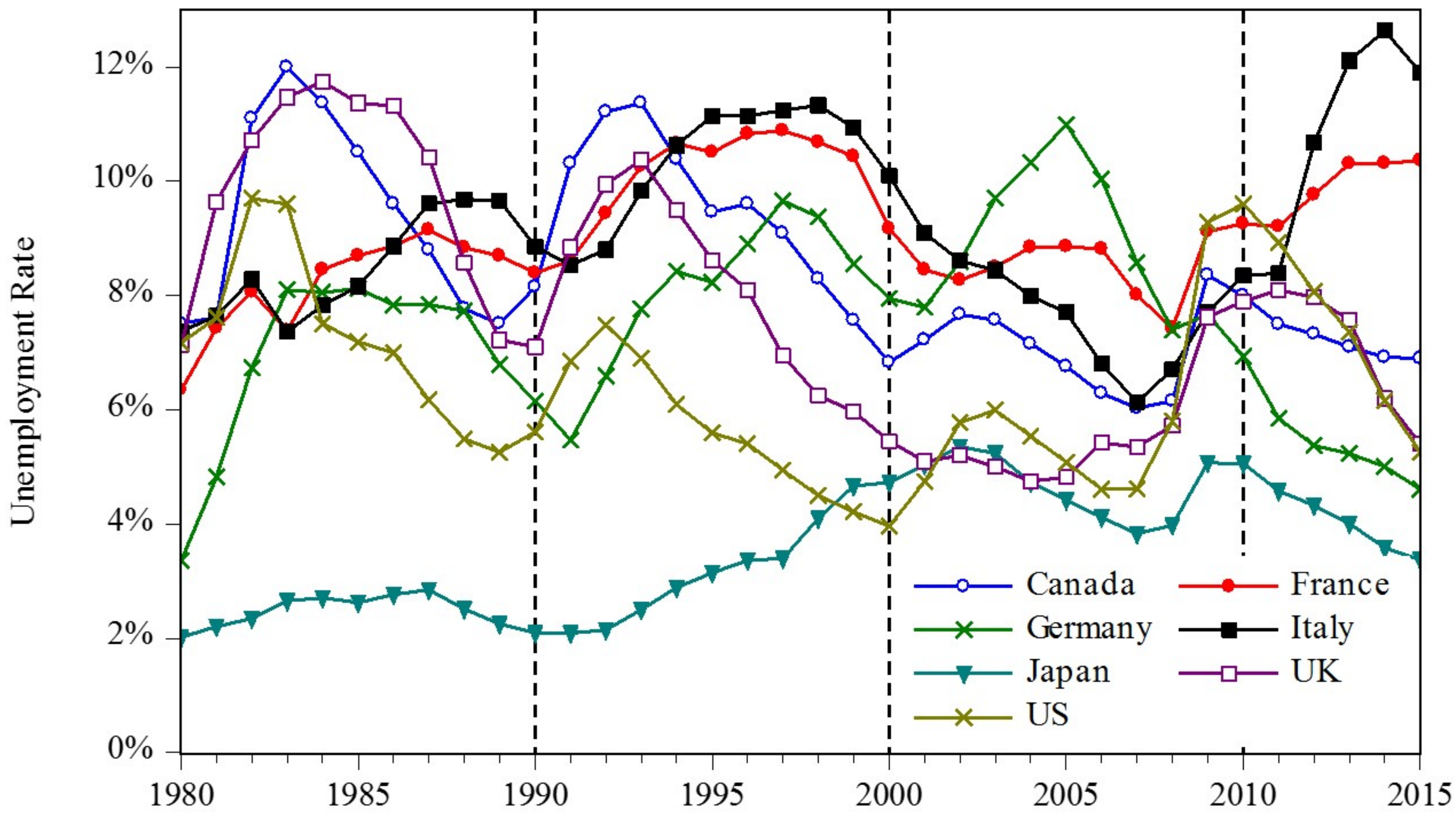

Figure A.2: Unemployment across G7 countries between 1980 and 2015. *Source*: Data from International Monetary Fund (2017a[21]).

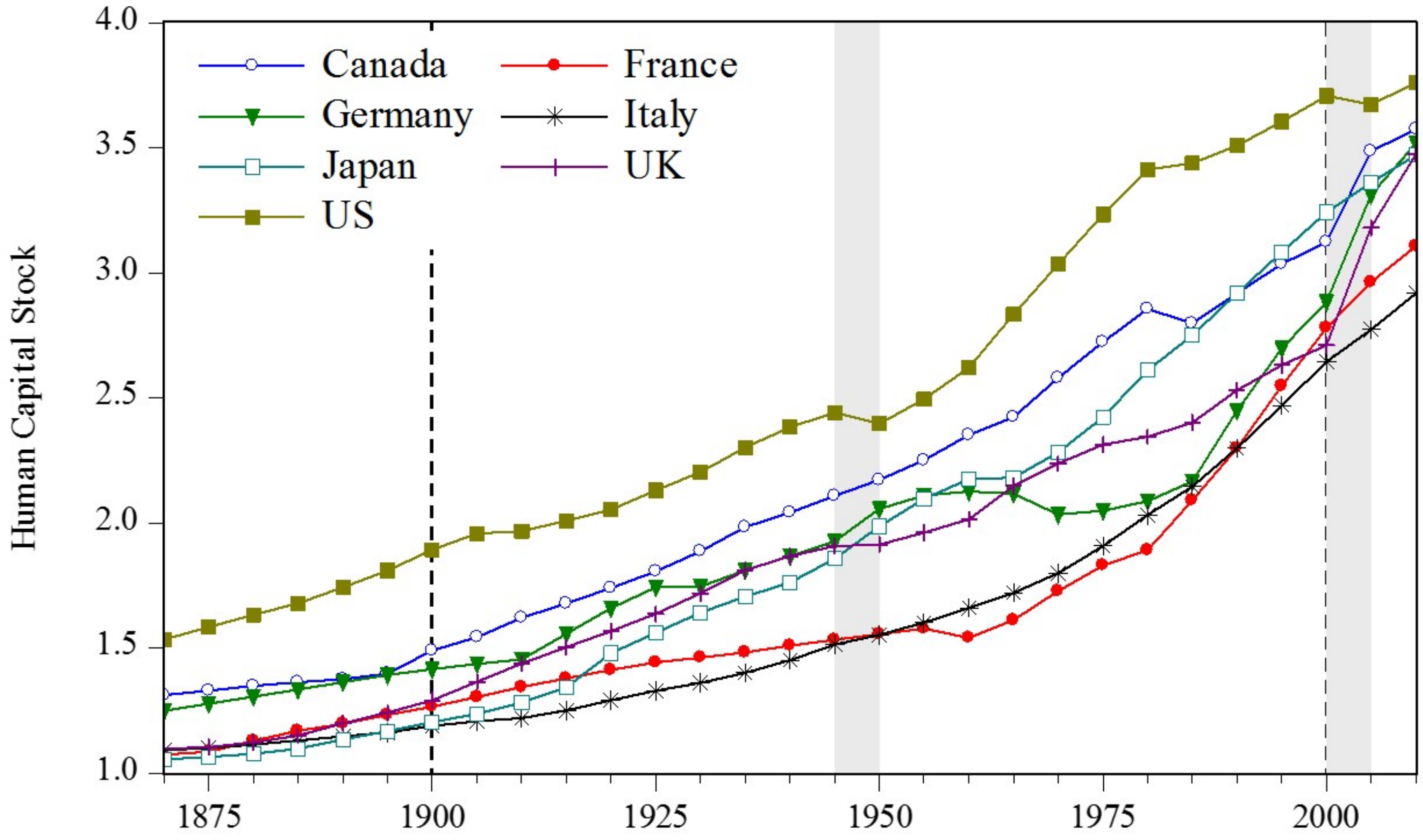

Figure A.3: Human capital stock across G7 countries. *Source*: Data set accompanying Lee and Lee (2016[27]).

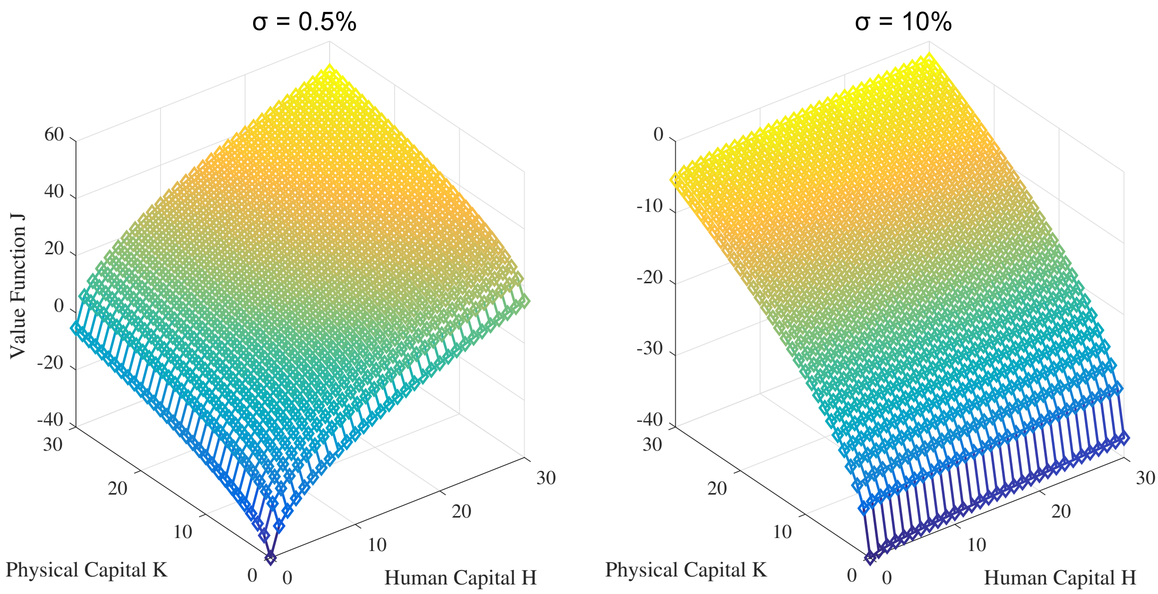

Figure A.4: Value function $J(K,H)$ visualized using $\rho = 0.05$. The left panel is the case for the virtually deterministic environment, whereas the right panel is for the stochastic environment. You can see that the lowering σ improves the value function $J(K,H)$ or the household welfare.

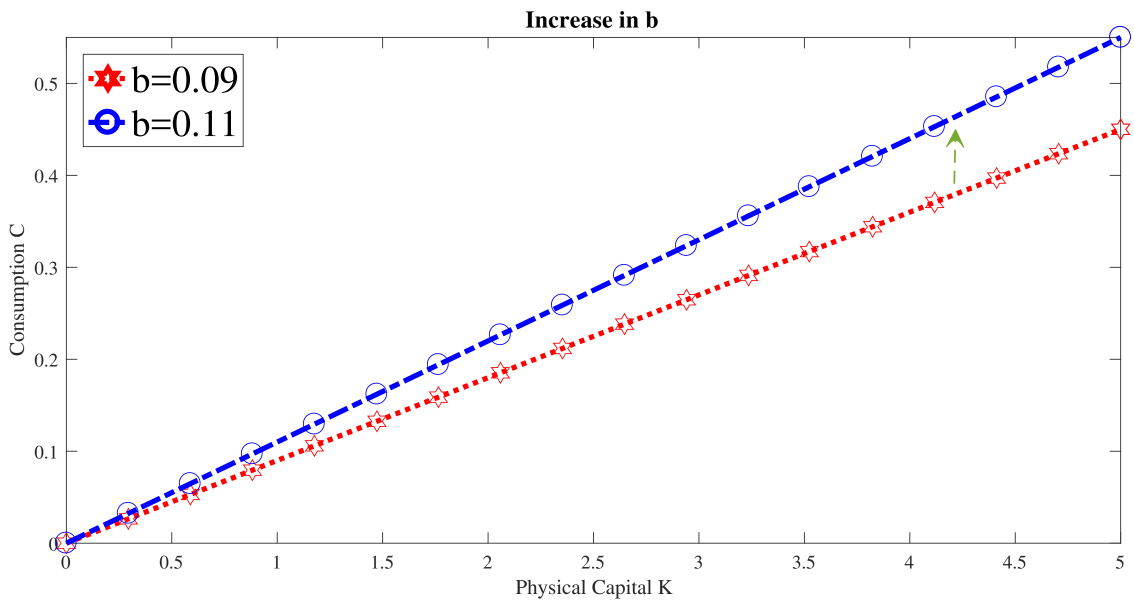

Figure A.5: Policy function. Given K, C gets stronger as b increases.

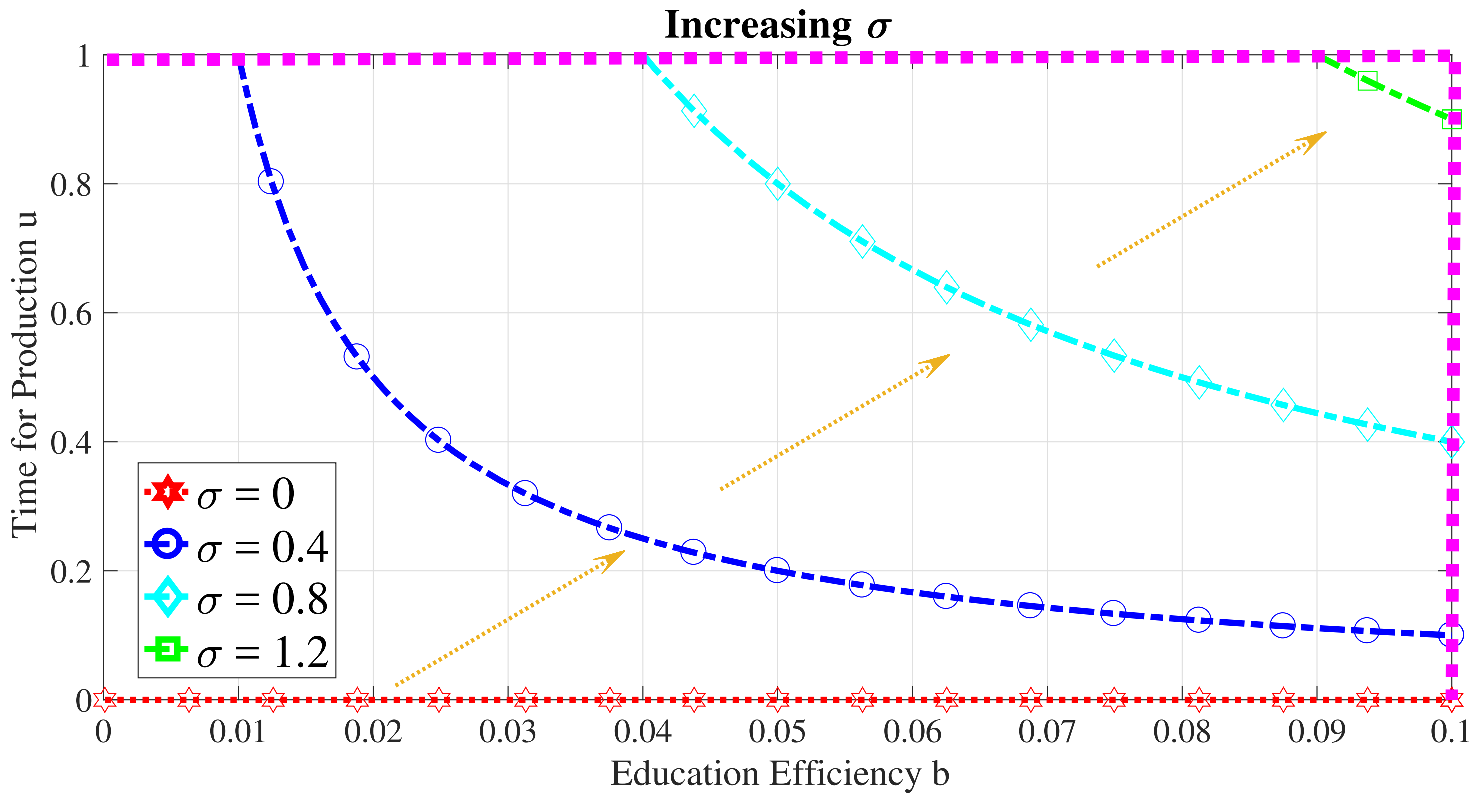

Figure A.6: Increase in σ given b. As σ increases from $\sigma = 0.0001$ to $\sigma = 0.4$, $\sigma = 0.8$, and to $\sigma = 1.2$, time devoted to production u increases, with u getting harder to lie in the unit interval. When $b = 0.1$, σ lies in the unit interval, so long as σ is not too large.