Analytical Foundations of the Uncertainty-Welfare Nexus: Technology Shocks and Stochastic Accumulation of Human Capital

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Abstract

The welfare of risk-averse agents is supposed to be reduced by higher uncertainty. Several recent studies, however, point out that agents may enjoy higher welfare in fluctuating economies than in their nonstochastic counterparts. I address this possibility by finding the closed-form solution to the stochastic Uzawa-Lucas model featuring correlated Brownian motion and many Poisson jump processes. For a moderate degree of uncertainty, I analytically show that higher uncertainty can improve welfare. I demonstrate that, if two stochastic processes are correlated, then there exists a hump-shaped relationship between welfare and two sorts of uncertainty.

Keywords: Technology, Human Capital, Welfare, Stochastic Growth, Analytical Solution
JEL: I31, J24, O41

1. Introduction

Can risk-averse agents enjoy higher welfare in fluctuating economies than in their nonstochastic counterparts? The answer is supposed to be no, since they do not like risk. Recent several studies of Cho et al. (2015), Lester et al. (2014), and Xu (2017), nonetheless find that the answer might be yes: more fluctuation may in fact improve the welfare of agents. Along the lines of their studies, I analytically show that, for a moderate degree of uncertainty, higher uncertainty *can* improve welfare. However, as usual, it is reduced if the size of shocks is large. I verify it by finding the closed-form solution to the stochastic Uzawa-Lucas model that features correlated Brownian motion and many Poisson jump processes. I demonstrate that a hump-shaped relationship between welfare and two sorts of uncertainty emerges, if two stochastic processes are correlated. A combination of technology and the accumulation of human capital enables it. Even if agents are risk averse, their welfare may be improved with higher uncertainty, under the circumstances.

Uncertainty is often considered something bad to the economy[1]. It seems quite natural to presume that higher uncertainty is always associated with declines in economic activity,

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[1] Throughout, I avoid drawing a distinction between uncertainty, risk, volatility, and shock. As apparent in the literature, these words are used interchangeably to mean the same thing. There is no cost of following this standard practice.

thereby reducing welfare. It immediately follows that uncertainty must, at any rate, be completely wiped out by sound policies, so that welfare ameliorates. Recent studies of Cho et al. (2015), Lester et al. (2014), and Xu (2017), however, cast doubt about the conventional positive relationship between uncertainty and welfare. They all show that, in point of fact, higher uncertainty may *improve* welfare, because purposeful agents can make use of uncertainty in their favor, if they can[2].

Their findings are important, because by intuition, the welfare-maximizing outcome seems to be feasible only in the complete absence of shocks. They overturn this under some conditions. If they are true, welfare-improving policies aiming for eliminating uncertainty may result in unanticipated repercussions. They may rather *deteriorate* the welfare of agents by getting rid of favorable shocks for the use of purposeful agents. Consequently, it is an absolute necessity for us to radically understand the condition under which larger shocks may improve welfare. Having the best grasp of that can capacitate policymakers to avoid the implementation of welfare-reducing policies that were supposed to *improve* the welfare of agents.

Undeniably, this emerging enterprise, including my paper, might be wide of the mark. Trusting our instincts, the welfare enhancement thanks to larger shocks would really be impossible, as shown by a great deal of study. Even then, it is not yet enough to jump to a final conclusion, unless more work on the emerging side is accumulated. As such, the first purpose of this paper is to uncover the new mechanism through which higher uncertainty can possibly improve welfare.

Specifically, I complement the above three papers in the framework of the stochastic two-sector optimal growth model of Uzawa (1965) and Lucas (1988), in which human capital is an explicit input for the production of final goods. Although the focus of Cho et al. (2015), Lester et al. (2014), and Xu (2017), are on the *household* side (such as its attitude toward risk and the intertemporal elasticity of substitution, IES), I instead focus my attention on the *production* side. I then find the *positive* relationship between welfare and uncertainty *without* varying the parameter(s) governing the household behavior.

Furthermore, unlike them, I *analytically* characterize the relationship between welfare and uncertainty throughout. For example, perturbation methods developed by Schmitt-Grohé and Uribe (2004), used by real business cycle (RBC) models of Cho et al. (2015) and Lester et al. (2014), are local. Thus, if the economy is far from the nonstochastic steady state, they might perform poorly. On the other hand, the projection method used by Xu (2017) is global, and likely to provide much better approximation. However, a cold fact is that no approximation can be more accurate than the closed-form solution[3].

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[2]These papers are inspired by the literature on the (Lucas) welfare cost of business cycles. Studies in that tradition, which always find increasing uncertainty is welfare-reducing, are thoroughly reviewed in these three papers. Therefore, to save space, I focus on reviewing these three papers. They miss Bramoullé and Treich (2009), though. Bramoullé and Treich (2009) analyze the global pollution problem under uncertainty. They show that, in a static model with n-player game, increasing uncertainty can reduce emissions and improve the welfare (of pollutants, not of consumers). See also Lucas (2003) for an excellent survey of this literature, including his original 1987 work.

[3]Though I need one parameter restriction commonly used in the literature, as discussed below.

Moreover, Xu (2017) uses the simple AK model for its tractability. But the problem with AK models is that they do not make an explicit distinction between technological progress and capital accumulation (Aghion and Howitt, 2009, p.66), and between physical and human capital[4]. Making it introduces the greater difficulty of obtaining the closed-form solution. It however enables me to find a hump-shaped interrelation between welfare and two sorts of shocks, at the cost of tractability. Despite that, I strive to obtain the analytical solution, which is impossible in some cases in Xu (2017). Over and above numerical simulation being unnecessary, the analytical solution allows us to inspect the mechanism in a transparent way. Without it, finding the threshold value at which the effect of uncertainty on welfare switches would not be possible. As such, an explicit solution will play a key role in this paper.

That being said, I now turn to the second contribution of my paper: obtaining the closed-form solution to the stochastic growth models in continuous time. It has been well-known that finding the analytical solution to stochastic growth models is often impossible[5]. For example, Wälde (2011a) surveys the state of the literature on *one*-sector stochastic growth models. He finds that, even for the simple one-sector AK model with logarithmic (or constant relative risk aversion, CRRA) utility and linear production technology following the stochastic process, the analytical solution is usually latent[6]. This intractability limits the applicability of continuous-time methods under uncertainty.

At the same time, the breakthrough has finally breezed in. Bucci et al. (2011) first find the closed-form solution to the *two*-sector optimal growth model of Uzawa-Lucas with CRRA utility, the generalized Cobb-Douglas production function à la Mankiw et al. (1992), and technology following an exogenous geometric Brownian motion process, with *two* parameter restrictions. Subsequently, Hiraguchi (2013) proves that their solution does not satisfy the optimality conditions and the closed-form solution is available with *one* parameter restriction only. In addition, in a similar vein, Marsiglio and La Torre (2012a, 2012b) obtain the closed-form solution to the stochastic Uzawa-Lucas model in which population dynamics is stochastic, with *two* parameter restrictions. As such, they all show that continuous-time methods under uncertainty can be appreciated not only in one-sector, but also in the *two*-sector stochastic growth models[7].

However, they all miss the hallmark of stochastic growth models: a full inquiry into a link between uncertainty and welfare. As Turnovsky (1997) and Turnovsky (2000) demonstrate, with the closed-form solution at our disposal, we are able to scrutinize the impact of stochastic elements on the welfare of agents. This is possible *only* in stochastic models, be-

[4]Solow (2005, p.10) also persuasively criticizes AK models as follows: "I thought they [AK models] were uninteresting story, in the sense that they more or less assumed what they purported to prove, and also misleading guides to policy, in that they made something look easy that is in fact very difficult."

[5]On this point, see, for instance, Merton (1975, p.384), Dixit and Pindyck (1994, p.78), and Turnovsky (2000, p.580).

[6]See also the introduction of Bucci et al. (2011) for a list of many studies that have tried finding the explicit solution to stochastic growth models.

[7]Hiraguchi (2009), Naz et al. (2016), and Chaudhry and Naz (2018) also find the closed-form solution to the Uzawa-Lucas model in continuous time, though in the deterministic environment.

cause deterministic models can speak nothing about effects of shocks on welfare. Therefore, I use the explicit solution to inspect a connection between welfare and uncertainty.

Withal, although four studies cited above deal only with a geometric Brownian motion process, I study the stochastic Uzawa-Lucas model highlighting a *combination* of controlled Brownian motion and many Poisson jump processes that is investigated in Wälde (2011a). Hiraguchi (2014) also obtains the explicit solution to the one-sector stochastic Ramsey model with leisure in which technological progress is driven by a mixture of a geometric Brownian motion and many Poisson jump processes[8].

In stark contrast to Bucci et al. (2011), Marsiglio and La Torre (2012a, 2012b), and Hiraguchi (2013), I consider the *controlled* diffusion process for the accumulation of human capital. With this formulation, as emphasized in Cho et al. (2015) and Lester et al. (2014), agents have endogenous choice (that is, either to work in the production sector or to learn in the human capital sector), which may allow them to make use of shocks in their favor. But, as we will see, unlike Cho et al. (2015) and Lester et al. (2014), it turns out that allowing purposeful agents to endogenously make their decisions is *not* the necessary condition for the positive relationship between uncertainty and welfare. I find that it essentially requires the *correlation* between two diffusion processes.

This suggests that the condition for the positive relation found by Cho et al. (2015) and Lester et al. (2014) can be necessary *only* in RBC models. It cannot be necessary in other models, or, at best, can be sufficient. In this study, in response to higher uncertainty, households tend to spend more time in learning, thereby accelerating the accumulation of human capital. For a moderate degree of uncertainty, this positive effect of uncertainty dominates its standard negative impact due to risk aversion, thus leading to a net welfare gain. However, when shocks are large, the usual negative effects outweigh the positive impact, resulting in a net welfare loss. As a consequence, there exists a hump-shaped relationship between uncertainty and welfare.

Summing up, the purpose of this paper is to show that, under some conditions, higher uncertainty *can* improve welfare by obtaining the closed-form solution to the stochastic Uzawa-Lucas model with two correlated Brownian motion and many Poisson jump processes, hence contributing to the literature on the uncertainty-welfare nexus and on obtaining the closed-form solution to stochastic (Uzawa-Lucas) endogenous growth models.

The paper is organized as follows. Section 2 sets up the stochastic Uzawa-Lucas model with two correlated diffusion and jump processes. I then find the closed-form solution and use it to examine the link between welfare and uncertainty. Section 3 considers another source of human capital uncertainty. Concluding remarks appear in Section 4.

[8]In the context of endogenous growth models, Poisson jump processes have been frequently used, for instance, in the creative destruction or Schumpeterian growth model of Aghion and Howitt (1992). They are also theoretically studied by Sennewald and Wälde (2006) and Sennewald (2007) in detail. Steger (2005) compares a Brownian motion process with a Poisson jump process in the AK model.

2. The Model

In this section, I develop the stochastic Uzawa-Lucas model in which both technology and the accumulation of human capital follow stochastic processes. Throughout, I assume that the total number of workers L equals unity, so that the per capita terms are equivalent to the aggregate terms. This assumption is also made in Bucci et al. (2011) and Hiraguchi (2013) as it greatly simplifies the analysis, and as population growth is not substance of this paper.

A representative household is endowed with one unit of time and uses all of that. It either works or learns. There is no other use of time. Let $u(t) \in (0,1)$ denote the fraction of time spent working to produce final goods $Y(t)$. Correspondingly, $1-u(t)$ represents the fraction of time spent learning. The amount of leisure is fixed exogenously, so there is no choice about it[9].

2.1. Capital Accumulation and Household

The accumulation of human capital $H(t)$ is stochastically governed by

$$dH(t) = b(1-u(t))H(t)dt - \delta_h H(t)dt + \sigma_h H(t)dz_h + \sum_{i=1}^{N} \beta_i H(t)dq_{it} \tag{1}$$

where dz_h is the increment of a Brownian motion (or Wiener) process such that the mean $\mathcal{E}(dz_h) = 0$ and variance $\mathcal{V}(dz_h) = dt$. $\sigma_h > 0$ is the diffusion coefficient of human capital (if $\sigma_h = 0$, then we would recover the deterministic limit). $b > 0$ is an exogenous parameter that indicates how efficient human capital accumulation is. $\delta_h \in (0,1)$ is its depreciation rate. More $u(t)$ mirrors less $1-u(t)$, thereby reducing the growth rate of human capital. There are N independent Poisson jump processes q_{it} with the arrival rate λ_i and jump size $\beta_i > -1$. I assume that the initial stock of human capital $H(0) = H_0 > 0$ is given, so that $H(t) > 0$ for all t with probability 1.

Note that the stochastic process (1) is the controlled diffusion process, that is, it contains one of key control variables in the Uzawa-Lucas model, $u(t)$, in the drift term of the diffusion process. Bucci et al. (2011), Marsiglio and La Torre (2012a, 2012b), and Hiraguchi (2013) assume that technological progress is stochastic, while the process of human capital accumulation is deterministic ($\sigma_h = 0$ and no jumps), which is at odds with the empirical literature such as Hartog et al (2007). A lack of human capital uncertainty has also been frequently pointed out by, for instance, Levhari and Weiss (1974) and Krebs (2003). In response to them, I assume that the process is stochastic, not deterministic. Treating the control variable in the drift term of a stochastic process makes it much harder to obtain the closed-form solution. However, as we will see, it enables us to examine the interplay between uncertainty and welfare in a transparent way.

[9] An explicit incorporation of leisure makes it extremely difficult to obtain the analytical solution. Therefore, I abstract from it. See Ladrón-De-Guevara et al. (1999) and Solow (2000) for the deterministic Uzawa-Lucas model with leisure. No study has found the closed-form solution to the stochastic Uzawa-Lucas model with leisure.

Next, the economy-wide resource constraint is

$$dK(t) = \underbrace{A(t)^{\gamma}(u(t)H(t))^{\alpha}K(t)^{1-\alpha-\gamma}}_{\equiv Y(t)}dt - C(t)dt - \delta_k K(t)dt \tag{2}$$

where $K(t)$ is physical capital and $\gamma > 0$. $\alpha \in (0, 1)$ represents the human capital share of income in the generalized Cobb-Douglas production function originally used by Mankiw et al. (1992), which is also used by Bucci et al. (2011) and Hiraguchi (2013). These imply $\alpha + \gamma \in (0, 1)$. $\delta_k \in (0, 1)$ is the depreciation rate of physical capital. $C(t)$ denotes consumption of the final good. The initial stock of physical capital $K(0) = K_0 > 0$ is given as well.

$A(t)$ is technology. Following Hiraguchi (2014), I assume that $A(t)$ follows *not only* a geometric Brownian motion process as in Bucci et al. (2011) and Hiraguchi (2013), but also a jump process - that is, a *mixture* of a geometric Brownian motion process and many Poisson jump processes:

$$dA(t) = \mu A(t)dt + \sigma_a A(t)dz_a + \sum_{j=1}^{n} \beta_j A(t)dq_{jt} \tag{3}$$

where $\mu > 0$ denotes the exogenous growth rate of technology. $\sigma_a > 0$ is the diffusion coefficient of technology. dz_a is, again, the increment of a Brownian motion process such that $\mathcal{E}(dz_a) = 0$ and $\mathcal{V}(dz_a) = dt$. There are n independent Poisson jump processes q_{jt} whose arrival rate is λ_j and the jump size is $\beta_j > -1$. I assume that the initial stock of technology $A(0) = A_0 > 0$ is also given, so that $A(t) > 0$ for all t with probability 1.

Unlike previous studies, the key assumption I make here is that two diffusion processes are *correlated*, i.e. $(dz_h)(dz_a) = \eta dt$, with η being the correlation coefficient of dz_h and dz_a. η will play a vital role in anatomizing the relationship between uncertainty and the welfare of agents[10]. Note that, technically, if $\eta = 0$, $\sigma_h = 0$, and no Poisson jumps, then my model recovers that of Hiraguchi (2013).

Preferences of a representative household are given by the standard CRRA utility:

$$E \int_0^{\infty} e^{-\rho t} \frac{C(t)^{1-\phi} - 1}{1 - \phi} dt \tag{4}$$

where E is the mathematical expectation operator with respect to the information set available to the representative household. $\rho > 0$ is the subjective discount rate of that, i.e., the rate at which utility is discounted. $\phi > 0$ is the index of relative risk aversion (and $1/\phi$ is IES). When future consumption is uncertain, a larger ϕ makes future utility gain smaller, raising the value of additional future consumption.

[10]Some empirical studies point out the importance of the *interaction* between human capital and technological progress. Although Jones (2016) dismisses the importance of human capital in economic growth, Hojo (2003) estimates that human capital is significant for growth *if* it promotes technological progress. Recently, Madsen (2014) and Cinnirella and Streb (2017) argue that, the true value of human capital is well appreciated in conjunction with technological progress. Therefore, this assumption seems plausible.

In sum, a representative household maximizes its expected utility (4) subject to the law of motion for physical capital (2) and two stochastic processes for human capital accumulation (1) and technological progress (3).

2.2. Optimization and Closed-Form Solution

To solve this stochastic optimization problem, let $J(K, A, H)$ denote the value function. Then, the corresponding Hamilton-Jacobi-Bellman (HJB) equation is

$$\rho J = \max_{C(t),u(t)} \Bigg(\frac{C(t)^{1-\phi} - 1}{1-\phi} + \frac{E}{dt} \Bigg(J_K dK + J_A dA + J_H dH + \frac{J_{AA}(dA)^2}{2} + \frac{J_{HH}(dH)^2}{2} + J_{HA}(dH)(dA) \\ + \sum_{i=1}^{N} (J(K, A, (1+\beta_i)H) - J(K, A, H))dq_{it} + \sum_{j=1}^{n} (J(K, (1+\beta_j)A, H) - J(K, A, H))dq_{jt} \Bigg)$$

where $J_X = \partial J/\partial X$, $J_{XX} = \partial^2 J/\partial X^2$, and $J_{XY} = \partial J/\partial X \partial Y$ for variables X and Y. First-order conditions are

$$C = J_K^{-\frac{1}{\phi}} \tag{5}$$

and

$$u = \frac{K}{H} \left(\frac{J_K \alpha A^{\gamma}}{J_H b K^{\gamma}} \right)^{\frac{1}{1-\alpha}} \tag{6}$$

Substituting these first-order conditions (5) and (6) into the above HJB equation, after some algebra, we arrive at

$$0 = \frac{\phi}{1-\phi} J_K^{\frac{\phi-1}{\phi}} - J_K \delta_k K - \frac{1}{1-\phi} - \rho J(K, A, H) + \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{1}{1-\alpha}} b^{\frac{\alpha}{\alpha-1}} J_K^{\frac{1}{1-\alpha}} J_H^{\frac{\alpha}{\alpha-1}} A^{\frac{\gamma}{1-\alpha}} K^{\frac{1-\alpha-\gamma}{1-\alpha}} \\ + J_H b H - J_H \delta_h H + J_A \mu A + \frac{\sigma_a^2}{2} J_{AA} A^2 + \frac{\sigma_h^2}{2} J_{HH} H^2 + J_{HA} \eta \sigma_h \sigma_a H A \\ + \sum_{i=1}^{N} \lambda_i (J(K, A, (1+\beta_i)H) - J(K, A, H)) + \sum_{j=1}^{n} \lambda_j (J(K, (1+\beta_j)A, H) - J(K, A, H))$$

With this HJB equation, out task now is to "guess and verify" the closed-form representation of the value function $J(K, A, H)$ as there yet exists no algorithm to figure it out. Unfortunately, the explicit solution is unfeasible to this problem. As is well known, in general, the partial differential equations like the HJB equation cannot be solved analytically. Nonetheless, it gets obtainable if we impose one parameter restriction. It can be summarized as follows.

Theorem 1. *Define* $\mathcal{S} \equiv -\sum_{i=1}^{N} \lambda_i((1+\beta_i)^{\alpha} - 1) - \sum_{j=1}^{n} \lambda_j \left((1+\beta_j)^{\gamma} - 1\right)$. *If we impose* *the following parameter constraint originally suggested by Xie (1991) and extensively used in the literature since then,*

$$\phi = 1 - \alpha - \gamma \tag{7}$$

then we can find the closed-form representation of the value function (that satisfies both the HJB equation and the transversality condition, or TVC) of the form

$$J(K, A, H) = \mathcal{X} K^{\alpha+\gamma} + \mathcal{Y} A^{\gamma} H^{\alpha} + \mathcal{Z} \tag{8}$$

where

$$\mathcal{X} \equiv \frac{1}{\alpha+\gamma}\left(\frac{1-\alpha-\gamma}{\rho + (\alpha+\gamma)\delta_k}\right)^{1-\alpha-\gamma}$$

$$\mathcal{Y} \equiv \frac{\mathcal{X}(\alpha+\gamma)}{b^{\alpha}}\left(\frac{1-\alpha}{\rho - \mu\gamma - \alpha b + \alpha\delta_h + \frac{\sigma_a^2}{2}\gamma(1-\gamma) + \frac{\sigma_h^2}{2}\alpha(1-\alpha) - \eta\alpha\gamma\sigma_a\sigma_h + \mathcal{S}}\right)^{1-\alpha} \tag{9}$$

$$\mathcal{Z} \equiv -\frac{1}{\rho(\alpha+\gamma)}$$

The corresponding expressions for control variables are

$$\begin{aligned} C &= (\mathcal{X}(\alpha+\gamma))^{-\frac{1}{1-\alpha-\gamma}} K \\ &= \frac{\rho + (\alpha+\gamma)\delta_k}{1-\alpha-\gamma} K \end{aligned} \tag{10}$$

and

$$\begin{aligned} u &= \left(\frac{(\alpha+\gamma)\mathcal{X}}{b\mathcal{Y}}\right)^{\frac{1}{1-\alpha}} \\ &= \frac{\rho - \mu\gamma - \alpha b + \alpha\delta_h + \frac{\sigma_a^2}{2}\gamma(1-\gamma) + \frac{\sigma_h^2}{2}\alpha(1-\alpha) - \eta\alpha\gamma\sigma_a\sigma_h - \mathcal{S}}{b(1-\alpha)} \end{aligned} \tag{11}$$

Proof. See Appendix A[11].

2.3. Comments on Theorem

I in turn comment on the main points in Theorem 1.

[11]The condition for $u \in (0, 1)$ is lengthy and can easily be obtained by straightforward calculation.

2.3.1. Parameter Restriction and Value Function

The parameter restriction (7), originally proposed by Xie (1991), says that the risk aversion parameter equals the physical capital share of income. Whether it holds true in practice is still open debate, because the estimation of ϕ is a task of great difficulty. For example, on the one hand, Lucas (2003) claims that ϕ ranges from 1 (log utility) to 4, but on the other, Smith (2007) says that ϕ should be smaller than 1. Despite this, the restriction (7) has been widely used by a number of authors in order to obtain the closed-form solution to their model. Xie (1991, 1994), Rebelo and Xie (1999), Smith (2007), Bucci et al. (2011), Marsiglio and La Torre (2012a, 2012b), and Hiraguchi (2013), all use the restriction (7), and I follow them, as it allows us to inspect the underlying mechanism in the most transparent way.

At a first glance, imposing the parameter restriction may look problematic. But this is one of the major approaches in the literature on continuous-time methods under uncertainty. For example, Wälde (2011b, p.277) states that "For a much larger class of models - which are then standard models - closed-form solutions cannot be found...Economists then either go for numerical solutions...or they restrict the parameter set in a useful way. Useful means that with some parameter restriction, value functions can be found again and closed-form solutions are again possible."

One may wish to back down the parameter restriction (7) and thereby the closed-form representation (8), and instead resort to, say, the value function iteration or (implicit) finite-difference method. Although that can be another approach, they too have some shortcomings. For instance, there is no guarantee that the value function would converge to the "true" one. Or, even if it does, without the analytical solution, it would be hard to see what is drawing one's findings. As we will see, the closed-form solution at our disposal is essential for perspicuously understanding the relationship between uncertainty and welfare. Therefore, the real problem is not the parameter restriction itself: what really matters is to evaluate which "cost" - the cost of imposing the parameter restriction and the cost of numerical approximation - is higher. Here, following many studies cited above, I proceed on grounds that the former is lower.

Equation (8) is the closed-form representation of the value function that will be used in the welfare analysis below. We can see that physical capital and the product of human capital and technology are separable. Note that, again, when $\sigma_h = \eta = 0$ and there is no Poisson uncertainty, the value function (8) completely coincides with that of Hiraguchi (2013, equation (29)). As he notes, the non-separability implies that endogenous growth comes from a fusion of technological progress and the accumulation of human capital. This observation is consistent with empirical studies of Madsen (2014) and Cinnirela and Streb (2017).

2.3.2. Control Variables and Expected Growth Rate

Equation (10) tells us that the consumption-capital ratio is constant. It seems a bit at odds that the optimal level of consumption is neither dependent of human capital stock $H(t)$ nor of technology $A(t)$. However, Wälde (2011a) and Hiraguchi (2013) also observe this property. Since it is found in the one-sector stochastic growth model of Smith (2007) as well, the optimal level of consumption appears to linearly depend on physical capital stock.

Equation (11) says that the time spent in working is constant as well, again consistent with Hiraguchi (2013). You can see that u involves key parameters relevant to stochastic processes. Here, the most important difference between this paper and Hiraguchi (2013) is that, the effect of diffusion coefficients σ_h and σ_a on u is *indeterminate*. Specifically, u is always increasing in σ_a in Hiraguchi (2013, p.137). However, in sharp contrast to the previous studies, because there are *two* diffusion processes that are *correlated*, the effect of one shock is dependent on the other, and can be indeterminate.

This has remarkable implications for the growth rate of human capital, which is the quintessence of the Uzawa-Lucas model. Due to stochasticity, it is not possible to compute its *actual* growth rate. However, it is still possible to calculate its *expected* growth rate, since u turns out to be constant. In fact, one can show that the *expected* growth rate of human capital $\mathcal{G}_h$ is given by

$$\mathcal{G}_h \equiv \mathcal{E}\left(\frac{\dot{H}}{H}\right) = \frac{b - \rho - \delta_h + \mu\gamma - \frac{\sigma_a^2}{2}\gamma(1-\gamma) - \frac{\sigma_h^2}{2}\alpha(1-\alpha) + \eta\alpha\gamma\sigma_a\sigma_h - \mathcal{S} + (1-\alpha)\sum_{i=1}^{N}\lambda_i\beta_i}{1-\alpha}$$

where $\dot{H} \equiv dH/dt$. First, note that, in the absence of technological progress (μ), depreciation, and uncertainty terms, the sign of $\mathcal{G}_h$ depends exclusively on the relative size of b and ρ. As discussed in Kuwahara (2017), it is the usual property of the deterministic Uzawa-Lucas model. It turns out that my model also has that property, as it should.

Second, since η is one of the most important parameters in this paper, we have to understand its impact on expected growth rate. One can show that

$$\frac{\partial \mathcal{G}_h}{\partial \eta} > 0$$

thus, higher correlation raises the expected growth rate of human capital. To see why, notice that the proportion of time devoted to learning u is decreasing in η, as shown in (11). This means that higher η discourages people to work, or equivalently, encourages them to accumulate their new human capital. Therefore, since the accumulation of human capital is accelerated, the expected growth rate of human capital increases in response to higher correlation between two stochastic processes. Since this is the important mechanism through which parameters of interest affect major macroeconomic variables, I illustrate this point in Figure 1, using parameters listed in Table 1[12]. Below, I will explain the details of Figure 1.

[12]Following Mankiw et al. (1992, p.432), I set the human capital share $\alpha = 1/3$. For physical capital share, it has been commonplace in macroeconomics to assume that it is also 1/3. However, as Karabarbounis and Neiman (2014) document, the labor share is declining (or, put differently, physical capital share is rising) globally. Therefore, I set $\gamma = 0.27$ so that the physical capital share roughly equals 0.40, the value used by the continuous-time stochastic growth model of Ahn et al. (2017). $b = 0.11$ is the value when Barro and Sara-i-Martin (2004) use in simulating the Uzawa-Lucas model. I choose $\mu = 0.02$ and $\delta_k = \delta_h = 0.03$ again following Mankiw et al. (1992). $\sigma_h = \sigma_a = 0.01$ are purely for the illustrative purpose. Finally, following Caballé and Santos (1993) and Moll (2014), I set $\rho = 0.05$. Note that, unlike Xu (2017), the purpose of this paper is not the quantitative assessment of the policy.

At this stage, observe that the expected growth rate (left panel) and u (right panel) go in the opposite direction. In this paper, the allocation of time is the key to understanding the underlying mechanism[13].

Table 1: Baseline Parameters for Illustration

α	γ	b	μ	σ_h	σ_a	ρ	η	λ_i	λ_j	β_i	β_j	δ_k	δ_h
1/3	0.27	0.11	0.02	0.01	0.01	0.05	0	0	0	0	0	0.03	0.03

Third, what about the impact of human capital shock σ_h on $\mathcal{G}_h$? Unlike η, b, and μ, the sign is not determinate, for

$$\frac{\partial \mathcal{G}_h}{\partial \sigma_h} \begin{cases} > 0 & (\text{if } \sigma_h < \frac{\gamma\eta}{1-\alpha}\sigma_a) \\ < 0 & (\text{if } \sigma_h > \frac{\gamma\eta}{1-\alpha}\sigma_a) \end{cases} \tag{12}$$

In other words, the effect of higher human capital uncertainty on its expected growth rate is ambiguous. To understand this point, see Figure 1. The left panel shows the relationship between expected growth rate of human capital and the size of human capital uncertainty, while the right panel displays that between time allocation and the size of human capital uncertainty. Each line presented are indexed by the correlation coefficient η.

We begin with the case of no correlation $\eta = 0$. In this case, as represented by the line with circles in the right panel, higher uncertainty discourages people to accumulate their new human capital, as there is uncertainty in human capital sector. As a result, people spend more time in the production sector, leading to human capital contraction. Then, since the stock of human capital decreases, the expected growth rate of human capital is decreased, as shown by the line with circles in the left panel. The case of negative correlation $\eta < 0$ can be interpreted in a similar way.

The interesting case would be when $\eta > 0$, i.e. when two stochastic processes are positively correlated. In this case, we can see a *hump-shaped* relationship between expected growth rate and the size of shocks. To see why, remember that, as we saw above, the higher correlation strengthens the growth rate since it leads to more accumulation of human capital. Thus, in this case, there are two conflicting forces - the "accumulation" effect due to higher correlation and the "contraction" effect due to higher uncertainty. For a moderate degree of human capital uncertainty, the former effect outweighs the latter, hence the result is the accumulation of human capital, which raises its growth rate. However, beyond the threshold value at which the equality $\sigma_h = \gamma\eta\sigma_a/(1-\alpha)$ holds, the latter outweighs the former, resulting in the contraction of human capital. As such, the threshold value, *which is available thanks to the closed-form solution*, is the point where the relative "power" of two conflicting forces changes, thereby yielding a hump-shaped relationship between growth rate and the size of shocks.

Moreover, since we have

[13]By the same token, one can immediately see that $\mathcal{G}_h$ is raised by the more efficient accumulation of human capital ($\partial \mathcal{G}_h/\partial b > 0$) and higher growth rate of technology ($\partial \mathcal{G}_h/\partial \mu > 0$).

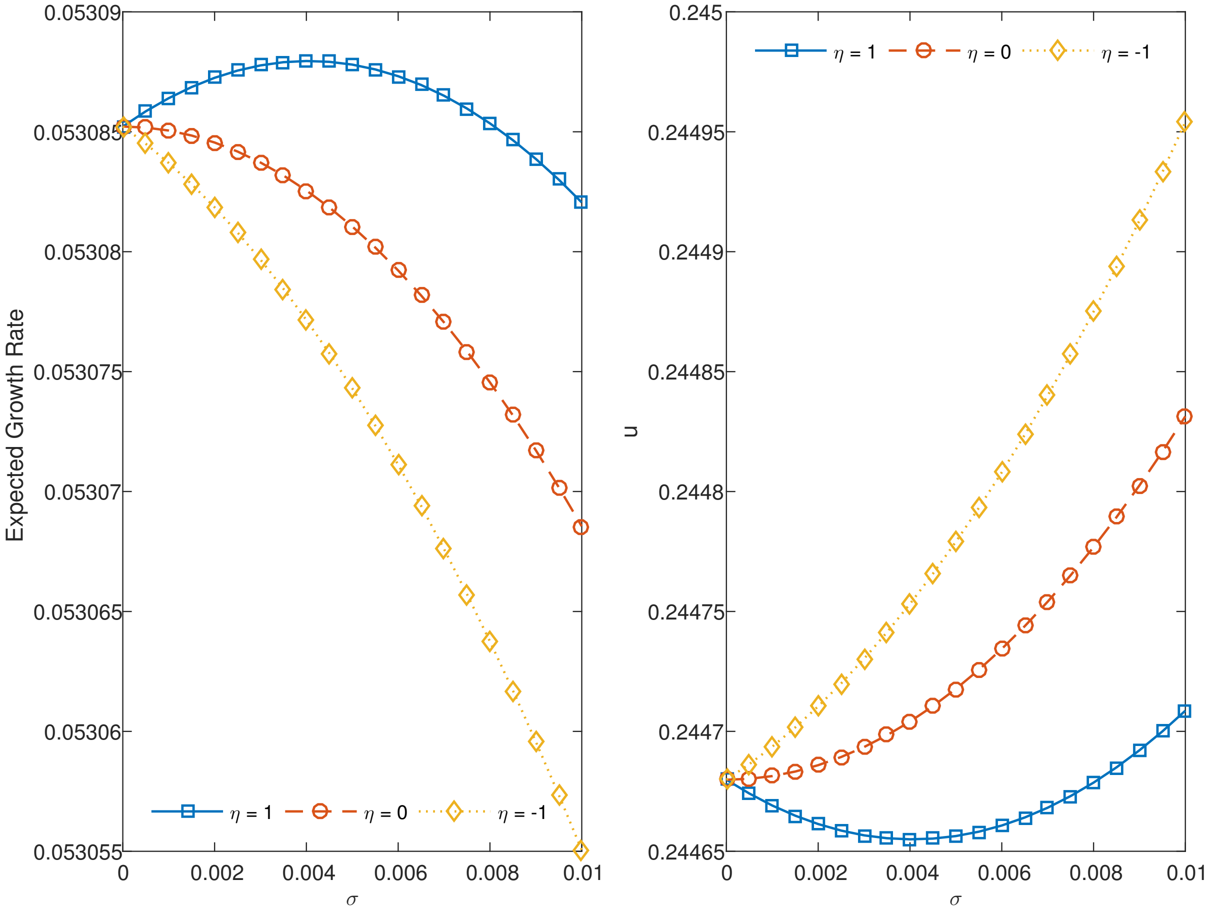

Figure 1: The relationship between the size of human capital shock σ_h and expected growth rate of human capital $\mathcal{G}_h$ (the left panel) and time allocation u (the right panel). Parameters are $\alpha = 1/3$, $\gamma = 0.27$, $b = 0.11$, $\mu = 0.02$, $\sigma_a = 0.01$, $\rho = 0.05$, $\delta_k = \delta_h = 0.03$, $\lambda_i = \lambda_j = \beta_i = \beta_j = 0$, $\eta = 0$ (line with circles), $\eta = -1$ (line with diamonds), and $\eta = 1$ (line with squares). When the correlation is smaller or equal to zero, higher uncertainty leads to human capital contraction, hence lower expected growth rate of human capital. When it is positive, there emerges a hump-shaped relationship, depending on the relative size of two conflicting forces. In principle, this figure also describes the case of technology shocks.

$$\frac{\partial u}{\partial \sigma_a}\begin{cases} > 0 & (\text{if } \sigma_h > \frac{1-\gamma}{\alpha\eta}\sigma_a) \\ < 0 & (\text{if } \sigma_h < \frac{1-\gamma}{\alpha\eta}\sigma_a) \end{cases} \tag{13}$$

one can easily gauge that the impact of technology shocks on the expected growth rate is again ambiguous, and that we would see the patterns described in Figure 1. As the underlying mechanism is completely the same, I refrain from repeating the same explanation above.

Finally, what about the impact of jumps on the expected growth rate? To begin with, the bigger size of both kinds of jump is *always* growth-enhancing ($\partial\mathcal{G}_h/\partial\beta_i > 0$ and $\partial\mathcal{G}_h/\partial\beta_j > 0$ for all i and j) since it accelerates the accumulation of human capital. Intuitively, when the return from the investment in human capital suddenly gets higher, it will be more attractive for people to invest in human capital. As a consequence, the larger jumps promote human capital accumulation, and in turn, increase the expected growth rate of human capital. The case of technology jump can be similarly considered (you can think about the mechanism described in Figure 1).

On the other hand, the implications of arrival rates would be more appealing, since for all i, we have

$$\frac{\partial \mathcal{G}_h}{\partial \lambda_i}\begin{cases} > 0 & (\text{if } \beta_i > 0) \\ < 0 & (\text{if } \beta_i \in (-1, 0)) \end{cases} \tag{14}$$

that is, effects of the arrival rate is indeterminate. The sign depends whether the size of jump is greater or less than zero. To get some intuition, see Figure 2. The left panel shows the relationship between expected growth rate of human capital $\mathcal{G}_h$ and its arrival rate λ_i, while the right panel shows that between time allocation u and the arrival rate λ_i. Each line presented is indexed by the jump size $\beta_i = 1\%, 0\%$, and -1%. The $\beta_i = 0\%$ line is plotted for the benchmark with no jumps. For this illustration, I assume that there is only one Poisson jump process ($N = 1$).

You can see that, when $\beta_i = 1\%$, the higher arrival rate causes people to spend more time in learning (line with squares, right panel). This is reasonable, because when the return from investment in human capital is likely to be positive and hence attractive, it would be desirable for the "event" to happen more frequently. Therefore, it leads to the accumulation of human capital, resulting in higher expected growth rate of human capital (line with squares, left panel). Unlike η, as there are not conflicting forces, this mechanism would be easier to understand. The case of $\beta_i = -1\%$ and of technology jump (β_j and λ_j) can be similarly explained.

Now, because we have analytically inspected and understood the mechanism through which uncertainty affects the expected growth rate of human capital, we are ready to examine the relationship between uncertainty and welfare. As the seminal paper of Barlevy (2004) shows, growth rates have close ties with welfare. As such, we can easily guess the effect of, for example, human capital uncertainty on welfare.

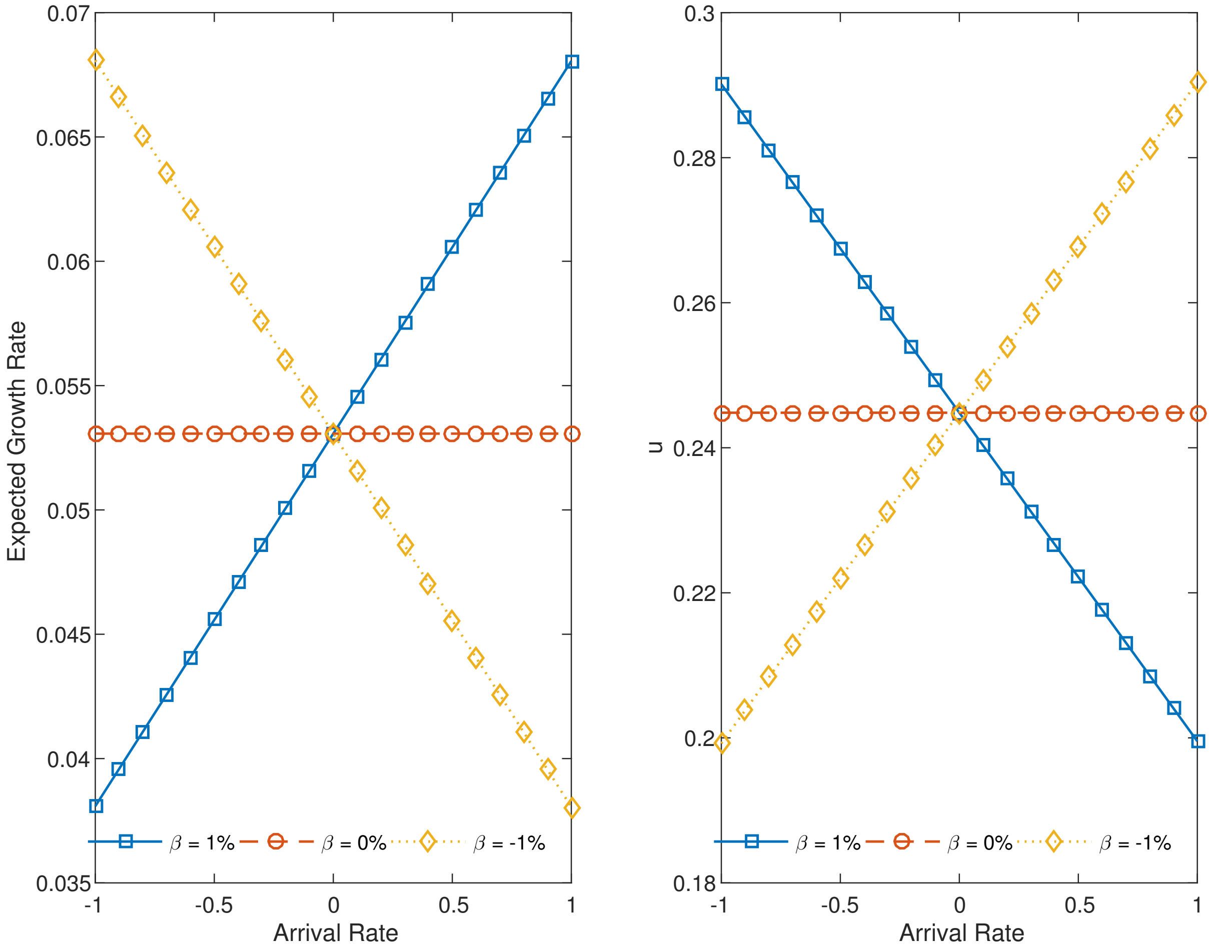

Figure 2: The relationship between the arrival rate of human capital λ_i and expected growth rate of human capital $\mathcal{G}_h$ (left panel) and time allocation u (right panel). Parameters are $\alpha = 1/3$, $\gamma = 0.27$, $b = 0.11$, $\mu = 0.02$, $\sigma_h = \sigma_a = 0.01$, $\rho = 0.05$, $\delta_h = \delta_k = 0.03$, $\beta_j = 0$, $\beta_i = 1\%$ (line with squares), $\beta_i = 0\%$ (horizontal line with circles), and $\beta_i = -1\%$ (line with diamonds). When the jump size is positive, the higher arrival rate encourages people to accumulate their new human capital, thereby enhancing the expected growth rate of human capital, and vice versa.

2.4. Welfare

In this subsection, I analyze the impact of uncertainty on welfare. As noted above, the underlying mechanism through which uncertainty affects welfare is quite similar to that described for the case of expected growth rate of human capital. Therefore, I will refrain from repeating the explanation already given above. Since we have the closed-form representation of the value function (8), the welfare analysis can be made in a transparent way; just partially differentiate the value function with respect to the parameter of interest[14].

First, we have

$$\frac{\partial J(K, A, H)}{\partial \eta} > 0 \tag{15}$$

that is, the higher correlation improves the welfare of agents. This happens through two channels. The first channel is obvious; since u is decreasing in η, the higher η accelerates the accumulation of human capital. Then, as the value function $J(K, A, H)$ is a function of the state variable H, it is improved. The second channel is through the constant $\mathcal{Y}$, (9). If you notice that $\mathcal{Y}$ is increasing in η, and that $\mathcal{Y}$ is in front of the product of technology and human capital (see (8)), you can see that the higher η improves the contribution of *both* state variables A and H (because of non-separability) to the welfare $J(K, A, H)$. In sum, increasing η improves H through u, and improves H and A through $\mathcal{Y}$. Through these two channels, the higher correlation of two diffusion processes turns out to be welfare-improving.

Second, we have

$$\frac{\partial J(K, A, H)}{\partial \sigma_h} \begin{cases} > 0 & (\text{if } \sigma_h < \frac{\gamma\eta}{1-\alpha}\sigma_a) \\ < 0 & (\text{if } \sigma_h > \frac{\gamma\eta}{1-\alpha}\sigma_a) \end{cases}$$

you may note that the condition for positive/negative signs coincides with that for the expected growth rate of human capital (see the partial (12)). Or, put differently, the condition for the positive sign for the expected growth rate is the same with that for the positive sign for the welfare of agents. I illustrate this point in Figure 3. Each line is again indexed by the correlation coefficient η. As you may have expected, when $\eta > 0$, there emerges a hump-shaped relationship between uncertainty and welfare.

To understand why, observe that in response to higher human capital uncertainty, households tend to spend more time in learning, thereby accelerating the accumulation of human capital. For a moderate degree of uncertainty, this positive effect of uncertainty dominates its standard negative impact due to risk aversion, thus leading to a net welfare gain. This is the reason why we see the positive relationship between uncertainty and welfare, as in Cho et al. (2015), Lester et al. (2014), and Xu (2017). However, when shocks to human capital accumulation are large enough, the usual negative effects outbalance the positive impact, resulting in a net welfare loss. As a consequence, there exists a hump-shaped relationship between uncertainty and welfare (as indicated by the line with squares, left panel).

[14]Following Turnovsky (1997, 2000), I use the value function $J(K, A, H)$ for welfare analyses.

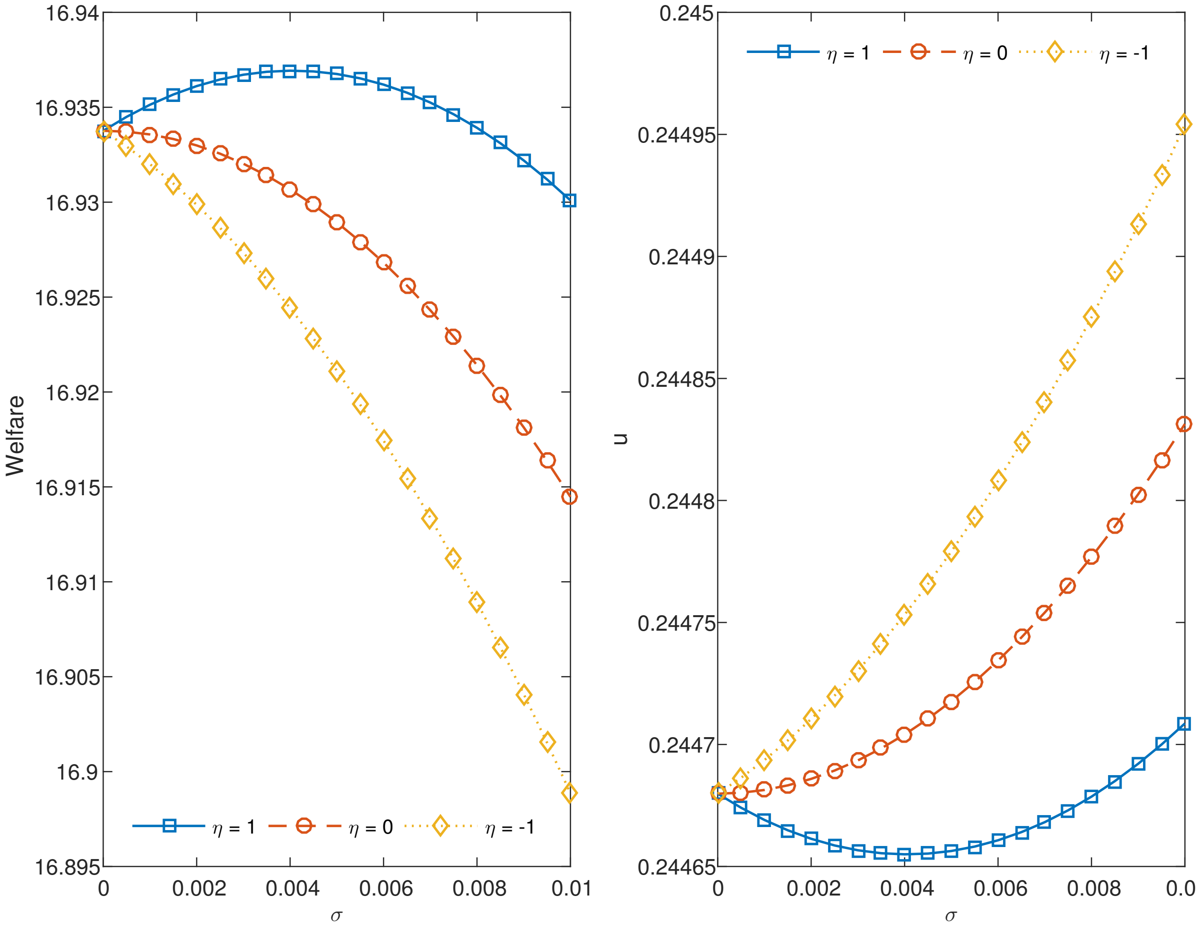

Figure 3: The relationship between the size of human capital shock σ_h and welfare $J(K, A, H)$ (the left panel) and time allocation u (the right panel). Parameters are $\alpha = 1/3$, $\gamma = 0.27$, $b = 0.11$, $\mu = 0.02$, $\sigma_a = 0.01$, $\rho = 0.05$, $\delta_k = \delta_h = 0.03$, $\lambda_i = \lambda_j = \beta_i = \beta_j = 0$, $\eta = 0$ (line with circles), $\eta = -1$ (line with diamonds), and $\eta = 1$ (line with squares). When the correlation is smaller or equal to zero, higher uncertainty leads to human capital contraction, thereby deteriorating the welfare of agents. On the other hand, when it is positive, there emerges a hump-shaped relationship between uncertainty and welfare. In principle, this figure also describes the case of technology shocks.

At this stage, you may think that this is possible because of the parameter restriction (7), which restricts $\phi < 1$, making people less risk-averse than "usual" ($\phi = 1$). It is not true, however. In fact, Lester et al. (2014) confirm that, in some cases, higher uncertainty improves welfare for a wider range of risk aversion parameter; from $\phi = 0.5$ to $\phi = 5.0$ (hence higher risk aversion). In other words, uncertainty in the *production* side can yield the necessary condition for the positive interrelation between welfare and uncertainty, and it *is* the contribution of this paper: this relationship can be found without varying the *household* side parameter ϕ, in sharp contrast to Cho et al. (2015), Lester et al. (2014), and Xu (2017).

For technology shocks, we have

$$\frac{\partial J(K, A, H)}{\partial \sigma_a} \begin{cases} > 0 & (\text{if } \sigma_h > \frac{1-\gamma}{\alpha\lambda}\sigma_a) \\ < 0 & (\text{if } \sigma_h < \frac{1-\gamma}{\alpha\lambda}\sigma_a) \end{cases}$$

As in the case of expected growth rate, Figure 3 is self-explanatory; there emerges a hump-shaped relationship between technology shocks and welfare, which in principle looks like the line with squares in the left panel of Figure 3. Although the underlying mechanism is the same, it is important to note that it overturns the finding of Cho et al. (2015) and Lester et al. (2014). They find that technology shocks *always* improve the welfare of agents. Here, however, even when two stochastic processes are positively correlated, if the size of technology shocks is large enough, technology shocks reduce welfare. The important lesson is that technology shocks are *not always* welfare-improving.

What about the welfare implications of jumps? As you may guess, it is easy to verify that the bigger jump of human capital and technology improves welfare. As before, it implies the higher return from the investment in human capital, making it attractive. This causes people to spend more time in learning, leading to the accumulation of human capital. Because welfare $J(K, A, H)$ is the function of the state variable H, welfare is improved. The additional welfare gain comes from the constant $\mathcal{Y}$.

As for the arrival rate, its impact on welfare is again indeterminate. It is illustrated in Figure 4. Just like the case of the expected growth rate of human capital, in response to higher/lower arrival rates, welfare and the proportion of time devoted to working go in the opposite direction when $\beta_i \neq 0$. When it is positive, higher arrival rates encourage people to spend more time in learning, hence accelerating the accumulation of human capital. In turn, because welfare is the function of the state variable H, the welfare of agents is improved. Again, there is additional welfare gain via $\mathcal{Y}$ defined in (9). The case of technology jump is too obvious to explain.

The findings of this section can be summarized as follows:

Proposition 1. *One parameter restriction commonly used by many previous studies makes it possible to find the closed-form solution to the stochastic Uzawa-Lucas model in which both the accumulation of human capital and technology are driven by the correlated Brownian motion and independent many Poisson jump processes. The higher correlation between two stochastic processes and bigger jumps always increase the expected growth rate of human capital and improve welfare. The effect of arrival rate on expected growth rate and*

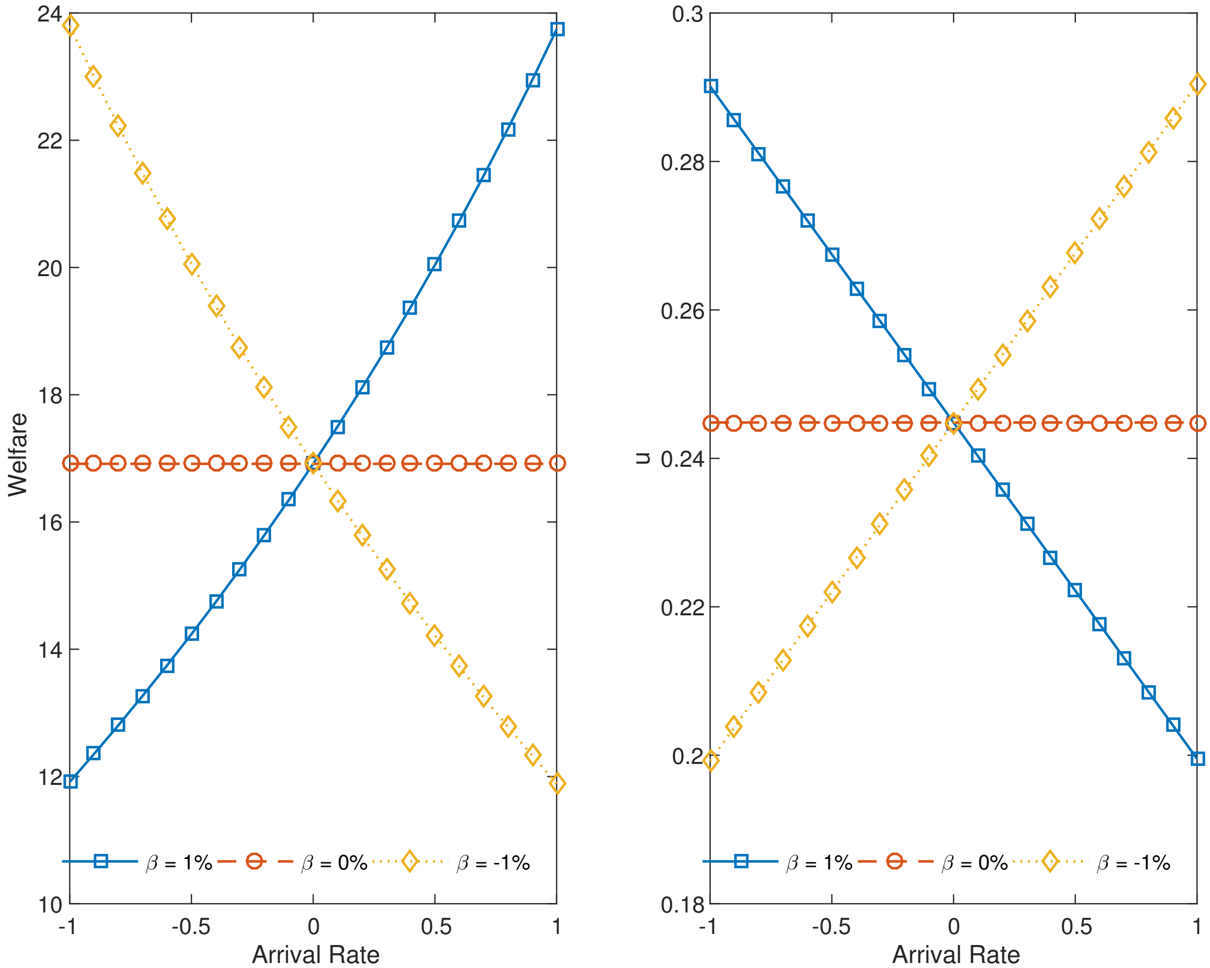

Figure 4: The relationship between the arrival rate of human capital λ_i and welfare $J(K, A, H)$ (left panel) and time allocation u (right panel). Parameters are $\alpha = 1/3$, $\gamma = 0.27$, $b = 0.11$, $\mu = 0.02$, $\sigma_h = \sigma_a = 0.01$, $\rho = 0.05$, $\delta_h = \delta_k = 0.03$, $\beta_j = 0$, $\beta_i = 1\%$ (line with squares), $\beta_i = 0\%$ (horizontal line with circles), and $\beta_i = -1\%$ (line with diamonds). When the jump size is positive, the higher arrival rate encourages people to accumulate their new human capital, thereby improving the welfare of agents, and vice versa.

welfare depends on whether jump size is positive or negative. When two stochastic processes are positively correlated, there exists a hump-shaped relationship between uncertainty and expected growth rate, and between uncertainty and welfare, as long as the size of shocks is small.

To recap, we have learned that the stochastic growth model - here, the two-sector optimal growth model of Uzawa-Lucas - has much to speak about the expected growth rate of human capital, and consequently, the welfare of agents. Despite that, these analyses are completely absent in Bucci et al. (2011), Marsiglo and La Torre (2012a, 2012b), and Hiraguchi (2013). Moreover, this is the first paper that analyzes the welfare-implication of *correlated* two diffusion processes and a *combination* of Brownian motion process and many Poisson jump processes, in the context of endogenous growth models. As long as one considers the only one diffusion process, or assumes that two processes are uncorrelated in order to simplify the analysis, the seemingly counterintuitive result can never be found.

3. Robustness

In the previous section, I assume that the *return* from the accumulation of human capital is stochastic. However, it may well be the case that it can be better described by the stochastic *depreciation* of human capital. Some authors such as Eaton (1981), Rebelo and Xie (1999), and Wälde (2011) analyze the stochastic growth model in which the depreciation of *physical* capital is stochastic. Since no study has explored the stochastic depreciation of *human* capital (at least in the context of the Uzawa-Lucas model), it would be worth analyzing. Therefore, in this section, I instead assume that human capital depreciation follows stochastic processes. This section may be regarded as the sensitivity analysis of the previous section. Does the different source of human capital uncertainty alter the findings? Is it still possible to find a closed-form solution?

3.1. Risky Human Capital

Now there are m independent Poisson jump processes q_{ht}:

$$dH(t) = b(1-u(t))H(t)dt - \delta_h dt - \sigma_h dz_h - \sum_{h=1}^{m} \beta_h H(t) dq_{ht} \tag{16}$$

where $\beta_h < 1$ is the jump size and λ_h is the associated arrival rate. The rest of the model remains unchanged. As a result, the corresponding HJB equation now reads

$$\rho J = \max_{C(t),u(t)} \Bigg(\frac{C(t)^{1-\phi}-1}{1-\phi} + \frac{E}{dt} \Bigg(J_K dK + J_A dA + J_H dH + \frac{J_{AA}(dA)^2}{2} + \frac{J_{HH}(dH)^2}{2} + J_{HA}(dH)(dA) + \sum_{j=1}^{n} (J(K,(1+\beta_j)A,H) - J(K,A,H)) dq_{jt} - \sum_{h=1}^{m} (J(K,A,(1-\beta_h)H) - J(K,A,H)) dq_{ht} \Bigg)$$

First-order conditions are still (5) and (6). Therefore, substituting these first-order conditions into the above HJB equation, after some algebra, we arrive at

$$\begin{aligned}\rho J =&\frac{\phi}{1-\phi}J_K^{\frac{\phi-1}{\phi}} - J_K\delta_k K - \frac{1}{1-\phi} + \left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{1}{1-\alpha}} b^{\frac{\alpha}{\alpha-1}} J_K^{\frac{1}{1-\alpha}} J_H^{\frac{\alpha}{\alpha-1}} A^{\frac{\gamma}{1-\alpha}} K^{\frac{1-\alpha-\gamma}{1-\alpha}}\\ &+ J_H bH - J_H\delta_h H + J_A\mu A + \frac{\sigma_a^2}{2}J_{AA}A^2 + \frac{\sigma_h^2}{2}J_{HH}H^2 + J_{HA}\eta\sigma_h\sigma_a HA\\ &+ \sum_{j=1}^{n}\lambda_j(J(K,(1+\beta_j)A,H) - J(K,A,H)) + \sum_{h=1}^{m}\lambda_h(J(K,A,(1-\beta_h)H) - J(K,A,H))\end{aligned}$$

With this new maximized HJB equation, our task is again to guess and verify the closed-form representation of the value function $J(K,A,H)$. Although the setting is slightly different, we can follow the same procedure. It can be summarized as follows.

Theorem 2. *Define $\mathcal{Q} \equiv -\sum_{j=1}^{n}\lambda_j\left((1+\beta_j)^{\gamma}-1\right) + \sum_{h=1}^{m}\lambda_h((1-\beta_h)^{\alpha}-1)$. If we again impose $\phi = 1-\alpha-\gamma$, then we can find the closed-form representation of the value function (that satisfies both the HJB equation and TVC) of the form*

$$J(K,A,H) = \mathcal{A}K^{\alpha+\gamma} + \mathcal{B}A^{\gamma}H^{\alpha} + \mathcal{C}$$

where $\mathcal{A} = \mathcal{X}$, $\mathcal{C} = \mathcal{Z}$, and

$$\mathcal{B} \equiv \frac{\mathcal{A}(\alpha+\gamma)}{b^{\alpha}}\left(\frac{1-\alpha}{\rho-\mu\gamma-\alpha b+\alpha\delta_h+\frac{\sigma_a^2}{2}\gamma(1-\gamma)+\frac{\sigma_h^2}{2}\alpha(1-\alpha)-\eta\alpha\gamma\sigma_a\sigma_h+\mathcal{Q}}\right)^{1-\alpha} \quad (17)$$

The corresponding expressions for the optimal level of consumption is still (10), while that for time spent in working u is

$$\begin{aligned}u &= \left(\frac{(\alpha+\gamma)\mathcal{A}}{b\mathcal{B}}\right)^{\frac{1}{1-\alpha}}\\ &= \frac{\rho-\mu\gamma-\alpha b+\alpha\delta_h+\frac{\sigma_a^2}{2}\gamma(1-\gamma)+\frac{\sigma_h^2}{2}\alpha(1-\alpha)-\eta\alpha\gamma\sigma_a\sigma_h+\mathcal{Q}}{b(1-\alpha)}\end{aligned} \quad (18)$$

Proof. See Appendix A[15].

[15] The condition for $u \in (0,1)$ can be again obtained by straightforward calculation.

3.2. Comments on Theorem and Expected Growth

Theorem 2 shows that it is still possible to find the closed-form solution even when depreciation of human capital is stochastic. However, this modelling choice does not alter the main predictions found in the previous section, as you can see in (17) and in (18). Moreover, the expected growth rate of human capital $\mathcal{G}_h^\delta$ is now given by

$$\mathcal{G}_h^\delta \equiv \mathcal{E}\left(\frac{\dot{H}}{H}\right) = \frac{b - \rho - \delta_h + \mu\gamma - \frac{\sigma_a^2}{2}\gamma(1-\gamma) - \frac{\sigma_h^2}{2}\alpha(1-\alpha) + \eta\alpha\gamma\sigma_a\sigma_h - \mathcal{Q} - (1-\alpha)\sum_{h=1}^{m}\lambda_h\beta_h}{1-\alpha}$$

which is not that different from the one in the previous section as well. Therefore, the analysis of uncertainty on expected growth rate of human capital and welfare is repetitive and thus redundant. Nonetheless, the robustness check of this section should be summarized as follows:

Proposition 2. *One parameter restriction used by many previous studies still allows us to find the closed-form solution to the stochastic Uzawa-Lucas model in which both the accumulation of human capital and technology are driven by the correlated Brownian motion and independent many Poisson jump processes (but now, the depreciation of, rather than the return from, human capital accumulation is stochastic). This modelling choice does not alter the major findings of the previous section.*

4. Concluding Remarks

The welfare of risk-averse agents is supposed to be deteriorated by more uncertainty. While it is confirmed by the majority of previous studies, few studies find the opposite: higher uncertainty may improve welfare. For a moderate degree of uncertainty, I show that, higher uncertainty *can* improve welfare when two stochastic processes are positively correlated. Conversely, when they are uncorrelated or negatively correlated, higher uncertainty is always detrimental to the welfare of agents. In this paper, in response to higher uncertainty, households tend to spend more time in learning, hence accelerating the accumulation of human capital. For a moderate degree of uncertainty, this positive effect of uncertainty dominates its standard negative impact due to risk aversion, thereby leading to a net welfare gain. However, when shocks are large, the usual negative effects outweigh the positive impact, resulting in a net welfare loss. As a consequence, there exists a hump-shaped relationship between uncertainty and welfare.

I characterize this seemingly impossible outcome by finding the closed-form solution to the stochastic Uzawa-Lucas model in which both technology and the accumulation of human capital follow correlated Brownian motion and many Poisson jump processes, with one parameter restriction only. Focusing on the *production* side, I demonstrate that, under the plausible parameter values, a hump-shaped relationship between welfare and two shocks unfolds. Therefore, the presence of uncertainty itself may not necessarily be welfare-reducing, under some conditions. As such, this paper has contributed to the literature on the uncertainty-welfare nexus and on obtaining the closed-form solution to stochastic (Uzawa-Lucas) endogenous growth models.

To summarize for the purposes of policy implications, Cho et al. (2015) and I both point out that the business cycle may be welfare-improving. While they conclude that "Policies that respond to shocks have to take account of the source of shocks and often will have the implication that the optimal policy will cause the economy to fluctuate more.", I instead argue that, what policymakers really must concentrate their attention is not on the source of shocks, but on the *interaction* and *size* of shocks.

If they observe only one shock, then they ought to eliminate it in order to improve welfare. However, if they observe two sorts of shocks, they have to take a pause before implementing the policy and see whether they are correlated. If correlated, the next step is to see how large each shock is. If it is large, policymakers should get rid of it. At the same time, if it is not too large, it is not necessarily optimal to wipe it out, because it may rather result in the *deterioration* of welfare. As such, the design of the optimal policy, in the presence of one or more shock(s), will never be easy.

The policy implication of this paper is *not* that uncertainty is "good." It is absolutely bad, and should be eradicated. Under some conditions, however, entirely eliminating uncertainty may rather reduce the welfare of purposeful agents. In sum, I analytically characterize theoretical guidelines for policy: there must be no such thing as the welfare-improving policy that unexpectedly deteriorates the welfare of agents.

The concluding caveat is that, as discussed in Lester et al. (2014), the above statements hold in *frictionless* models. Whether my findings, and those of related studies, generalize in models with frictions, such as New Keynesian models with price/wage stickiness, is uncertain. In that class of models, because the presence of nominal rigidities makes room for welfare-enhancing (monetary) policies, the exploration into a link between welfare and uncertainty will be harder. It would also be interesting to analyze whether we can find the closed-form solution when the utility function function is KPR utility of King et al. (1988), GHH utility of Greenwood et al. (1988), or more general one of Duffie and Epstein (1992) than the standard CRRA preferences. These are clearly challenging, but the possibility is probably here to stay.

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Appendix A. Guide to the Closed-Form Solution

In this Appendix, I show you how to find the functional form of the value function. The presentation here is based on the Appendix A of Bucci et al. (2011). I postulate the tentative value function of the form

$$J(K,A,H) = T_X K^{\Theta_1} + T_Y A^{\Theta_2} H^{\Theta_3} + T_Z$$

where T_X, T_Y, T_Z, Θ_1, Θ_2, and Θ_3 are unknown constants to be determined. Relevant partials are $J_K = \Theta_1 T_X K^{\Theta_1 - 1}$, $J_A = \Theta_2 T_Y A^{\Theta_2 - 1} H^{\Theta_3}$, $J_{AA} = \Theta_2(\Theta_2 - 1) T_Y A^{\Theta_2 - 2} H^{\Theta_3}$, $J_H = \Theta_3 T_Y A^{\Theta_2} H^{\Theta_3 - 1}$, $J_{HH} = \Theta_3(\Theta_3 - 1) T_Y A^{\Theta_2} H^{\Theta_3 - 2}$, and $J_{HA} = \Theta_2 \Theta_3 T_Y A^{\Theta_2 - 1} H^{\Theta_3 - 1}$. Substituting these into the HJB equation in Section 2, we get

$$\begin{aligned}
0 = {} & \frac{\phi}{1-\phi}(\Theta_1 T_X)^{\frac{\phi-1}{\phi}} K^{\frac{(\Theta_1-1)(\phi-1)}{\phi}} - \frac{1}{1-\phi} - \delta_k(\Theta_1 T_X) K^{\Theta_1} - \rho T_X K^{\Theta_1} - \rho T_Y A^{\Theta_2} H^{\Theta_3} - \rho T_Z \\
& + \mu \Theta_2 T_Y A^{\Theta_2} H^{\Theta_3} + \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{1}{1-\alpha}} b^{\frac{\alpha}{\alpha-1}} (\Theta_1 T_X)^{\frac{1}{1-\alpha}} (\Theta_3 T_Y)^{\frac{\alpha}{\alpha-1}} K^{\frac{\Theta_1-\alpha-\gamma}{1-\alpha}} A^{\frac{\gamma-\alpha\Theta_2}{1-\alpha}} H^{\frac{\alpha(\Theta_3-1)}{\alpha-1}} \\
& + b\Theta_3 T_Y A^{\Theta_2} H^{\Theta_3} - \delta_h T_Y \Theta_3 A^{\Theta_2} H^{\Theta_3} + \frac{\sigma_a^2}{2}\Theta_2(\Theta_2-1) T_Y A^{\Theta_2} H^{\Theta_3} + \frac{\sigma_h^2}{2}\Theta_3(\Theta_3-1) T_Y A^{\Theta_2} H^{\Theta_3} \\
& + \eta\sigma_h\sigma_a\Theta_2\Theta_3 T_Y A^{\Theta_2} H^{\Theta_3} + \sum_{i=1}^{N} \lambda_i \left((1+\beta_i)^{\Theta_3} - 1\right) T_Y A^{\Theta_2} H^{\Theta_3} + \sum_{j=1}^{n} \lambda_j \left((1+\beta_j)^{\Theta_2} - 1\right) T_Y A^{\Theta_2} H^{\Theta_3}
\end{aligned}$$

Setting $\Theta_1 = \alpha + \gamma$, $\Theta_2 = \gamma$, $\Theta_3 = \alpha$ and imposing the parameter constraint (7), we have, after rearranging,

$$\begin{aligned}
0 = {} & \left(\frac{\phi}{1-\phi}\left((\alpha+\gamma) T_X\right)^{\frac{\phi-1}{\phi}} - \rho T_X - \delta_k T_X (1-\phi)\right) K^{\alpha+\gamma} - \left(\frac{1}{1-\phi} + \rho T_Z\right) \\
& + \Bigg(-\rho T_Y + \mu\gamma T_Y + \alpha b T_Y - \alpha\delta_h T_Y + \eta\sigma_h\sigma_a\alpha\gamma T_Y + \frac{\sigma_a^2}{2}\gamma(\gamma-1) T_Y + \frac{\sigma_h^2}{2}\alpha(\alpha-1) T_Y \\
& + (1-\alpha)(b T_Y)^{\frac{\alpha}{\alpha-1}} \left((1-\phi) T_X\right)^{\frac{1}{1-\alpha}} + \sum_{i=1}^{N} \lambda_i \left((1+\beta_i)^{\alpha} - 1\right) T_Y + \sum_{j=1}^{n} \lambda_j \left((1+\beta_j)^{\gamma} - 1\right) T_Y A^{\gamma} H^{\alpha}
\end{aligned}$$

As this equation must be satisfied for all K, A, and H, this yields the closed-form representation of the value function (8) in the text. The proof of optimality conditions requires the verification theorem. See Appendix A of Bucci et al. (2011), Appendix B of Hiraguchi (2013), or Chang (2004, Chapter 4) for details in a Brownian motion process case. For the Poisson jump case, see Sennewald and Wälde (2006) or Sennewald (2007).

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