Exhaustible Resources, Welfare, and Technological Progress in the Stochastically Growing Open Economy

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Abstract

Will shortages of natural resources constrain economic growth? The answer seems yes, as the amount of natural resources on earth is finite. There can, however, be two excuses for this. First, the resource-saving technological progress can undo resource scarcity. Second, at the country level, countries can import necessary resources from other countries. This paper casts doubt on these two excuses. For technology, not all technological progress is resource-saving, and its arrival is unpredictable. For the import argument, at the global level, the world cannot make up for a shortage of natural resources by importing. To address these, I construct the open, stochastic two-sector growth model in continuous time. I use this model to answer the above question from the welfare viewpoint. I then analytically show that the answer is sensitive to the interaction between technology shocks and resource shocks. In some cases, I find that higher resource uncertainty accelerates the growth rate and improves welfare.

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1. Introduction

Countries with more natural resources are likely to achieve higher growth and can improve the welfare of household. This is because, in addition to physical and human capital, they can use natural resources such as coal, petroleum, and natural gas, to produce output. In parallel, we have to be conscious that most resources are exhaustible - they do not necessarily renew themselves at a sufficient rate. Unduly immoderate use of resources is not possible, at least from the long-run viewpoint, as the amount of natural resources on earth is fixed.

The hasty upshot of this is simple to imagine: at some point in the foreseeable future, we may use up all exhaustible resources on this globe. Unable to use resources in production, economic growth will slow down, and eventually we would have zero growth. With no growth afterward, economic welfare would deteriorate further and further. This is a worst-case scenario.

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If this standard narrative is coming true, the price of natural resources must go up, since we have "excess demand." However, take a look at Fig. 1. It shows the real price of industrial commodities, consisting of an equally-weighted basket of aluminum, coal, copper, lead, iron ore, and zinc, deflated by the consumer price index (initial value is normalized to 100). Over the 20th century, we see that the price of resources has *fallen*. Between 1900 and 2000, the index had approximately decreased from 100 to 20, a factor of one-fifth, albeit the energy crisis in 1970s. Therefore, the standard narrative fails for this period.

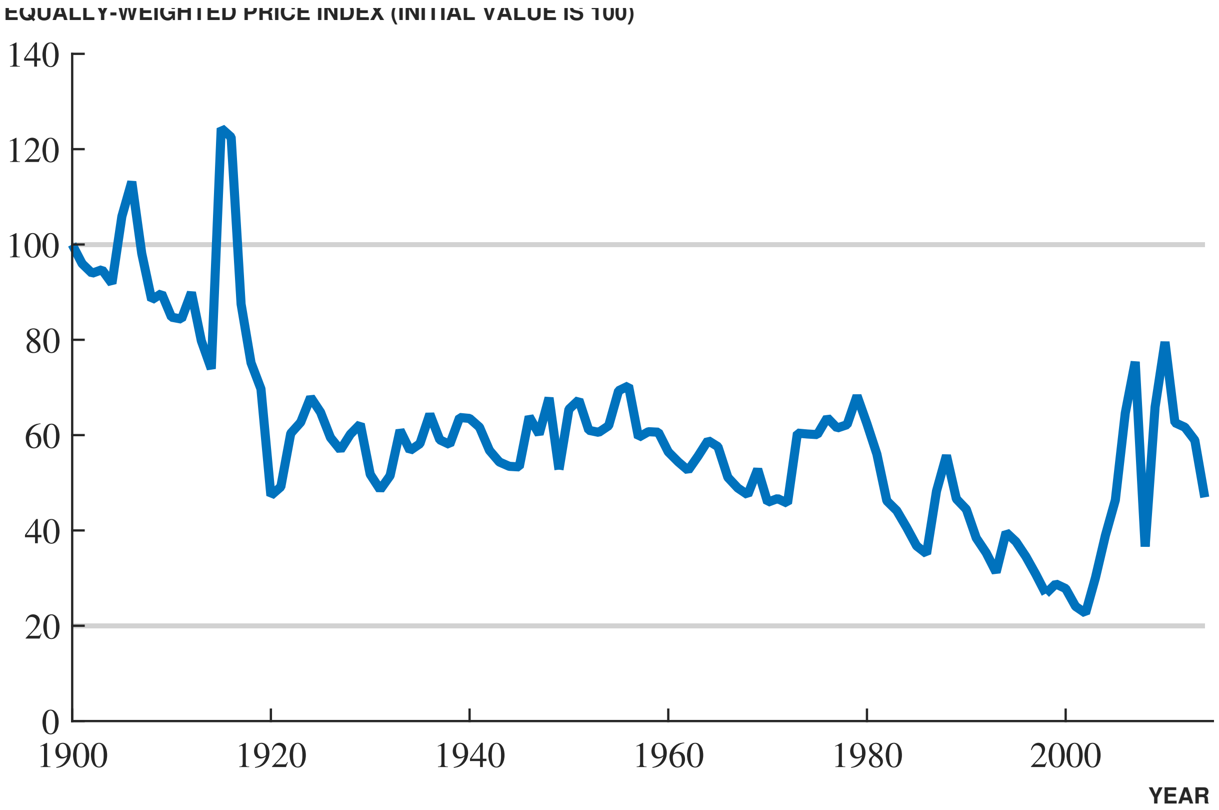

Figure 1: The real price of industrial commodities, consisting of an equally-weighted basket of aluminum, coal, copper, lead, iron ore, and zinc, deflated by the consumer price index. Source: Jones (2016, p.31).

But, since 2000, the price index have seemed to revert back to around 60, the average value between 1920 and 1980, and displayed the large fluctuation. At this stage, it is not possible to predict exactly whether the price will go up or go down in the foreseeable future. Nonetheless, increases in price index in 21st century may signal the scarcity of resources; we may be approaching the depletion of resources. If so, will growth slow down due to the constraint posed by resource scarcity, as predicted by an AK model with natural resources (Aghion and Howitt, 2009, Ch.16)[1]?

[1]On the other hand, Aghion and Howitt (2009, Ch.16) show that, even in the presence of exhaustible

Despite this concern, amazingly, this topic is absent in the masterly survey of Acemoglu (2009)[2]. This may reflect the view that the depletion of resources are not urgent concern, if not negligible. For example, Dasgupta and Heal (1974) argue that technological progress (i.e. the discovery of new substitutes) will make previously essential exhaustible resources *in*essential. On that account, as long as the process repeats, the depletion of resources will not pose a destructive problem. Another view is that, as Weil (2013) puts, countries can make up for any resources they lack by simply importing them from abroad. Truly, if a resource-poor country needs petroleum in production, it can import it from other countries. Due to the decline of transport costs over the last decades, the cost of replenishment via trade has been not great. This allows countries to avoid problems that would arise out of a scarcity of natural resources.

But these two views are not strong enough to claim that finite nature of natural resources can be avoided for an indefinite period of time. On the first view, technological progress that substitutes today's essential resources may not happen, and it is the reason why Dasgupta and Heal (1974) use the stochastic model in which the arrival date of the discovery is uncertain. In the absence of the discovery of substitutes, their logic does not hold. On the second view, it is valid at the *country* level, but not at the *global* level. Unlike a single country, the world cannot make up for a shortage of natural resources by importing. Again, the amount of resources on earth is not infinite.

Some studies have explored the implications of resource scarcity for economic growth. Solow (1978) analyzes this linkage using the CES (constant elasticity of substitution) production function[3]. Focusing on the elasticity of substitution between resources and other inputs (physical capital and labor), he concludes that resource scarcity would not pose a serious problem from the empirical point of view. Cheviakov and Hartwick (2009) augment the Solow (1956) model by incorporating exhaustible resources as an additional input. They show that the higher rate of depreciation of physical capital destroys the economy, whereas it can be avoided by strong technological progress. Vita (2007) extends the human-capital based endogenous growth model of Lucas (1988) by considering the substitutability between exhaustible resources and secondary materials (i.e. the manufactured material that has already been used at least once, and may be used again after recycling). He argues that varying substitutability affects economic growth rate during the transitional path.

None of the above studies has explored uncertainty involved in considering the resources. However, it is recognized that resource growth dynamics is in part stochastic[4]. In a partial equilibirum model, Pindyck (1984) examines the impact of resource uncertainty (modelled by stochastic processes). Interestingly, he finds that effects of higher resource fluctuation on the extraction rate is ambiguous; higher resource uncertainty can have the positive, zero, or negative influence, depending on the specification of the function governing the stochastic

resources, growth can be sustained in the creative destruction (or Schumpeterian) growth model. Romer (2012, Sec.1.8) describes why technological progress would make it possible to sustain growth, even in the presence of the resource depletion and land in the framework of neoclassical growth models.

[2]The lack of growth-resource linkage is also pointed out by Solow (2009).

[3]See also Solow (1974) for more on technical aspects of the growth-resource nexus.

[4]See Clark (1979), Pindyck (1984), and a number of references cited therein.

resource growth dynamics. Despite the insights that can be obtained by considering uncertainty associated with resource growth dynamics, recent growth-resource papers such as Vita (2007) and Cheviakov and Hartwick (2009) ignore it.

It is unfortunate that interesting insights have been found, but they are obtained in a separate field. This motivates me to develop the coherent, growth framework in which key elements - technological progress, resource, and its uncertainty - are put in one place, thereby making it possible to study how they interact each other. To this end, I extend the deterministic two-sector Uzawa (1965) - Lucas (1988) model with exhaustible resources, recently developed by Neustroev (2014). Specifically, I present the stochastic Uzawa-Lucas model in which technological progress and resource dynamics are driven by the correlated stochastic process. Besides, I consider the simple, global economy setting, so that there would be no room to import resources "from abroad." It allows me to partially answer the question posed above.

Those being said, I now turn to the second contribution of this paper; closed-form solution to stochastic growth models. Stochastic growth models, pioneered by Brock and Mirman (1972), have been the cornerstone in macroeconomics. They have made it possible to study the interaction between growth and uncertainty, and also led to the development of the real business cycle theory. Modern macroeconomic models are primarily based on these models, and they can be used to quantitatively examine how macroeconomic aggregates respond to exogenous shocks such as technology shocks.

One shortcoming of stochastic growth models, however, is their intractability. To a large extent, because of uncertainty, the model is analytically harder to solve than their deterministic counterparts, and they are often solved numerically around the steady state. Moreover, a case of closed-form solutions is by and large limited. For example, in order to obtain the closed-form solution, some authors use logarithmic utility function, instead of the more general constant relative risk aversion (CRRA) preferences for the household side. Or, for the firm side, production technology has been limited to linear or the so-called AK type, instead of Cobb-Douglas technology[5].

This being said, the breakthrough of Bucci et al (2011) is substantial: they first find the closed-form solution to the continuous-time version of the two-sector optimal growth model of Uzawa (1965) and Lucas (1988) in which technological progress follows an exogenous Brownian motion process. Although Bucci et al (2011) assume the CRRA preference and generalized Cobb-Douglas production technology à la Mankiw et al (1992), without ignoring depreciation of physical and human capital, they successfully discover the explicit solution with *two* parameter restrictions, one of which has been standard since Xie (1991, 1994). Economically, they uncover that the larger technology shock reduces the optimal level of consumption and time devoted to the production of final goods. The former is obtained by virtue of Jensen's inequality, while the latter is confirmed by numerical simulation.

Subsequently, Hiraguchi (2013) revisits Bucci et al (2011) and finds that, the closed-

[5]The other approach is to abstract from the depreciation of capital. See the introduction of Bucci et al (2011) and Wälde (2011a) for a list of studies that have tried getting the explicit solution to stochastic growth models.

form solution is, in fact, available with *one* parameter restriction of Xie (1991, 1994) only. Moreover, Hiraguchi (2013) proves that the value function of Bucci et al (2011) does not satisfy the optimality conditions, and unveils the new correct value function. It shows that technology shocks, de facto, have *nothing* to do with the optimal level of consumption and *increase* time devoted to the final goods production, both of which are in sharp contrast to Bucci et al (2011). Following the lead of them, Marsiglio and La Torre (2012a, 2012b) also get the closed-form solution to the Uzawa-Lucas model with stochastic population dynamics driven by, again, a Brownian motion process[6].

Four papers on the stochastic Uzawa-Lucas model above are innovative in that they *analytically*, not numerically, throw new light on the mechanism through which technology/demographic shocks affect the macroeconomy in a transparent way. At the same time, they leave some common possibilities to be studied. First, they only consider a Brownian motion process. While this process is widely used (Turnovsky, 1997, 2000), it is not the only stochastic process used in macroeconomics.

For instance, another stochastic process of the Poisson jump process is also used extensively by some authors such as Aghion and Howitt (1992), Steger (2005), Sennewald and Wälde (2006), Sennewald (2007), and more recently, Brunnermeier and Sannikov (2014). Moreover, Wälde (2011a) finds the closed-form solution to the stochastic AK model with a *combination* of a Brownian motion process and many Poisson jump processes, while Hiraguchi (2014) does so in the stochastic Ramsey model with leisure. To string studies cited above together, it is worth exploring whether we can find the closed-form solution to the stochastic Uzawa-Lucas model with a mixed Brownian and many Poisson jump processes.

Summing up, the purpose of this paper is to examine the implications of the natural resource scarcity for the growth and welfare, by finding the closed-form solution to the open, stochastic Uzawa-Lucas model in which technological progress and resource growth dynamics are driven by stochastic processes.

The paper is organized as follows. Section 2 sets up the model and discusses insights obtained from the analysis. Concluding remarks appear in Section 3.

2. The Model

In this section, I develop the stochastic Uzawa-Lucas model in which both technology and the depletion of exhaustible resources follow stochastic processes. Suppose that the world economy consists of J countries, indexed by $j = 1, ..., J$. Throughout, I assume that J is large enough (so that each country is small relative to the rest of the world and thus it ignores its effect on world aggregates) and that the total number of workers in country j, L_j, equals unity in all countries (so that the per capita terms are equivalent to the aggregate terms in all countries).

[6] In virtue of a variety of mathematical techniques, Hiraguchi (2009), Naz et al (2016), and Chaudhry and Naz (2018) also find the closed-form solution to the Uzawa-Lucas model in continuous time, though in the deterministic setup.

The latter assumption is also made in the closed economy of Bucci et al (2011) and Hiraguchi (2013), as it greatly simplifies the analysis, and as population growth is not substance of their and my paper. Besides, we suppose that each country admits a representative household. It is endowed with one unit of time and uses all of that. It either works or learns. There is no other use of time. Let $u_j(t) \in (0,1)$ denote the fraction of time spent working to produce final goods $Y_j(t)$. Correspondingly, $1 - u_j(t)$ represents the fraction of time spent learning. The amount of leisure is fixed exogenously, so there is no choice about it[7].

2.1. Capital Accumulation, Resource, and Household

The human capital accumulation equation in country j, $H_j(t)$, is given by

$$dH_j(t) = b_j(1 - u_j(t))H_j(t)dt - \delta_{jh}H_j(t)dt \tag{1}$$

where $b_j > 0$ is an exogenous parameter that indicates how efficient human capital accumulation is. $\delta_{jh} \in (0,1)$ is its depreciation rate. Less $u_j(t)$ mirrors more $1 - u_j(t)$, thereby accelerating the growth rate of human capital. I assume that the initial stock of human capital $H_j(0) = H_{j0} > 0$ is given.

Next, the resource constraint takes the form

$$dK_j(t) = \underbrace{A_j(t)^{\alpha_j} K_j(t)^{\beta_j} (u_j(t)H_j(t))^{\gamma_j} (\vartheta_j \bar{S}(t))^{1-\alpha_j-\beta_j-\gamma_j}}_{\equiv Y_j(t)} dt - C_j(t)dt - \delta_{jk}K_j(t)dt \tag{2}$$

where $K_j(t)$ is physical capital and $\alpha_j \in (0,1)$. $\beta_j \in (0,1)$ represents the physical capital share of income and $\gamma_j \in (0,1)$ denotes the human capital counterpart in the generalized Cobb-Douglas production function à la Mankiw et al (1992), which is also used by Bucci et al (2011) and Hiraguchi (2013). These imply $\alpha_j + \beta_j + \gamma_j \in (0,1)$. $\delta_{jk} \in (0,1)$ is the depreciation rate of physical capital. $C_j(t)$ denotes consumption of the final good $Y_j(t)$. The initial stock of physical capital $K_j(0) = K_{j0} > 0$ is given as well.

$A_j(t)$ is technology in country j. Following Hiraguchi (2014), I assume that $A_j(t)$ follows *not only* a geometric Brownian motion process (as in Bucci et al. 2011 and Hiraguchi 2013), but also a jump process - that is, a *mixture* of a geometric Brownian motion process and many Poisson jump processes (the former goes on all the time, while the latter occurs infrequently):

$$dA_j(t) = \mu_j A_j(t)dt + \sigma_{ja}A_j(t)dz_{ja} + \sum_{n=1}^{N} \beta_{jn}A_j(t)dq_{jn} \tag{3}$$

where $\mu_j > 0$ denotes an exogenous growth rate of technology. $\sigma_{ja} > 0$ is the diffusion coefficient of technology (if $\sigma_{ja} = 0$, then we would recover the deterministic limit). dz_{ja} is

[7] An explicit incorporation of leisure makes it extremely difficult to obtain the analytical solution. Therefore, I abstract from it. See Ladrón-De-Guevara et al (1999) and Solow (2000) for the deterministic Uzawa-Lucas model with leisure. No study has found the closed-form solution to the stochastic Uzawa-Lucas model with leisure.

the increment of a Brownian motion (or Wiener) process such that the mean $E(dz_{ja}) = 0$ and variance $Var(dz_{ja}) = dt$. As changes in the process over any finite interval of time are normally distributed, a variance increases linearly with the time interval dt. There are also N independent Poisson jump processes q_{jn} à la Aghion and Howitt (1992), with the mean arrival rate λ_{jn} and a jump of size $\beta_{jn} > -1$. During a time interval of infinitesimal length dt, the probability that a jump will occur is given by $\lambda_{jn}dt$, and the probability that a jump will not occur is given by $1 - \lambda_{jn}dt$. In other words, $dq_{jn} = \beta_{jn}$ with probability $\lambda_{jn}dt$, while $dq_{jn} = 0$ with probability $1 - \lambda_{jn}dt$[8]. I assume that the initial stock of technology $A_j(0) = A_{j0} > 0$ is also given, so that $A_j(t) > 0$ for all t with probability 1.

$S_j(t)$ denotes the amount of exhaustible resources available in country j at time t, and $\bar{S}$ is the amount of the *global* stock of exhaustible resources (the bar indicates that the variable is measured on a global scale). ϑ_j is an exogenous parameter which denotes the share of resources country j can use in the production of final goods (hence $\sum_{j=1}^{J} \vartheta_j = 1$). Its law of motion is also stochastically governed by a combination of a geometric Brownian motion process and many Poisson jump processes:

$$dS_j(t) = -\nu_j S_j(t)dt + \sigma_{js} S_j(t) dz_{js} + \sum_{m=1}^{M} \beta_{jm} S_j(t) dq_{jm} \tag{4}$$

where $\nu_j > 0$ denotes the extraction rate of exhaustible resources. $\sigma_{js} > 0$ is the diffusion coefficient of exhaustible resources. dz_{js} is, again, the increment of a Brownian motion process such that the mean $E(dz_{js}) = 0$ and variance $Var(dz_{js}) = dt$. There are M independent Poisson jump processes q_{jm} whose arrival rate is λ_{jm} and the jump size is $\beta_{jm} > -1$. In the absence of Poisson jump terms, Eq. (4) is akin to the stochastic differential equation for resource dynamics in Pindyck (1980, Eq. 3) and Pindyck (1984, Eq. 7). I assume that the initial stock of exhaustible resources $S_j(0) = S_{j0} > 0$ is given as well, so that $S_j(t) > 0$ for all t with probability 1.

Unlike previous studies, the key assumption I make here is that two diffusion processes are *correlated*, that is, $(dz_{ja})(dz_{js}) = \eta_j dt$, with η_j being the correlation coefficient of dz_{ja} and dz_{js}. We will see that η_j will play a vital role in anatomizing the implications of natural resource scarcity for central issues of this paper. Note that, technically, if $\eta_j = 0$, $\sigma_{js} = 0$, and no Poisson uncertainty, then my model recovers that of Hiraguchi (2013).

Finally, preferences of a representative household in country j at time $t = 0$ are given by the standard constant relative risk aversion (CRRA) utility:

$$E \int_0^\infty e^{-\rho_j t} \frac{C_j(t)^{1-\phi_j} - 1}{1 - \phi_j} dt \tag{5}$$

where E is the mathematical expectation operator with respect to the information set available to a representative household. $\rho_j > 0$ is its subjective discount rate, that is, the rate at which utility is discounted. $\phi_j > 0$ is the index of relative risk aversion (and $1/\phi_j$ is

[8]See Chang (2004) for an in-depth treatment of a Brownian motion (Wiener) process. Dixit and Pindyck (1994, Ch.3) provide a lucid account of a Poisson jump process.

intertemporal elasticity of substitution). When future consumption is uncertain, a larger ϕ_j makes future utility gain smaller, raising the value of additional future consumption.

In sum, a representative household in country j maximizes its expected utility (5) subject to the law of motion for the accumulation of human capital (1) and for physical capital (2), to two stochastic processes for technological progress (3) and for exhaustible resources (4), and to $\sum_{j=1}^{J} K_j = \bar{K}$, $\sum_{j=1}^{J} H_j = \bar{H}$, and $\sum_{j=1}^{J} S_j = \bar{S}$.

2.2. Optimization and Closed-Form Solution

To solve this stochastic optimization problem in continuous time, let $V_j(K_j, A_j, H_j, S_j)$ denote the value function, that is, the function of relevant state variables. Then, the corresponding Hamilton-Jacobi-Bellman (HJB) equation is

$$\begin{aligned}\rho_j V_j = \max_{C_j(t),u_j(t)} \Bigg(& \frac{C_j(t)^{1-\phi_j}-1}{1-\phi_j} + \frac{E}{dt}\left(V_{jK}dK_j + V_{jA}dA_j + V_{jH}dH_j + V_{jS}dS_j + V_{jAS}(dA_j)(dS_j)\right. \\ & + \frac{V_{jAA}(dA)^2}{2} + \frac{V_{jSS}(dS)^2}{2} + \sum_{n=1}^{N}(V_j(K_j,(1+\beta_{jn})A_j,H_j,S_j) - V_j(K_j,A_j,H_j,S_j))dq_{jn} \\ & + \sum_{m=1}^{M}(V_j(K_j,A_j,H_j,(1+\beta_{jm})S_j) - V_j(K_j,A_j,H_j,S_j))dq_{jm}\Bigg)\end{aligned}$$

where $V_{jX} = \partial V_j/\partial X_j$, $V_{jXX} = \partial^2 V_j/\partial X_j^2$, and $V_{jXY} = \partial V_j/\partial X_j \partial Y_j$ for variables X_j and Y_j. As control variables are C_j and u_j, first-order optimal conditions are characterized by

$$C_j = V_{jK}^{-\frac{1}{\phi_j}} \tag{6}$$

and

$$u_j = \frac{1}{H_j}\left(\frac{\gamma_j V_{jK} A_j^{\alpha_j} K_j^{\beta_j} (\vartheta_j \bar{S})^{1-\alpha_j-\beta_j-\gamma_j}}{b_j V_{jH}}\right)^{\frac{1}{1-\gamma_j}} \tag{7}$$

Substituting these first-order conditions (6) and (7) into the above HJB equation, after some algebra, we arrive at

$$
\begin{aligned}
0 =& \frac{\phi_j}{1-\phi_j} V_{jK}^{\frac{\phi_j - 1}{\phi_j}} - V_{jK}\delta_{jk}K_j - \frac{1}{1-\phi_j} - \rho_j V_j(K_j, A_j, H_j, S_j) + V_{jH}H_j(b_j - \delta_{jh}) \\
&+ \left(\frac{1-\gamma_j}{\gamma_j}\right) \gamma_j^{\frac{1}{1-\gamma_j}} b_j^{\frac{\gamma_j}{\gamma_j - 1}} V_{jK}^{\frac{1}{1-\gamma_j}} V_{jH}^{\frac{\gamma_j}{\gamma_j-1}} A_j^{\frac{\alpha_j}{1-\gamma_j}} K_j^{\frac{\beta_j}{1-\gamma_j}} (\vartheta_j \bar{S})^{\frac{1-\alpha_j-\beta_j-\gamma_j}{1-\gamma_j}} + \mu_j A_j V_{jA} + \frac{\sigma_{ja}^2 V_{j_A A} A_j^2}{2} \\
&+ \frac{\sigma_{jh}^2 V_{jSS} S_j^2}{2} + V_{jAS}\eta_j \sigma_{ja}\sigma_{js} A_j S_j + \sum_{n=1}^{N} \lambda_{jn}(V_j(K_j, (1+\beta_{jn})A_j, H_j, S_j) - V_j(K_j, A_j, H_j, S_j)) \\
&+ \sum_{m=1}^{M} \lambda_{jm}(V_j(K_j, A_j, H_j, (1+\beta_{jm})S_j) - V_j(K_j, A_j, H_j, S_j))
\end{aligned}
\tag{8}
$$

where I use the "fact" that $E(dq_i) = \lambda_i dt$ for $i = n, m$ (see, for example, Sennewald and Wälde (2006) or Wälde (2011b)). With this maximized HJB equation (8), out task now is to "guess and verify" the closed-form representation of the value function $V_j(K_j, A_j, H_j, S_j)$ as there yet exists no algorithm to figure it out. However, as is well known, in general, the partial differential equations like the HJB equation cannot be solved analytically by hand (what is worse, we have *four* state variables). Unfortunately, one can prove that the explicit solution is unfeasible to this problem. Nonetheless, it gets obtainable if we impose one parameter restriction. It can be summarized as follows.

Theorem 1. *Define*

$$
\Theta_j \equiv \frac{\sigma_{ja}^2 \alpha_j(1-\alpha_j) + \sigma_{js}^2(\alpha_j + \beta_j + \gamma_j)(1-\alpha_j-\beta_j-\gamma_j)}{2} - \alpha_j \eta_j \sigma_{ja}\sigma_{js}(1-\alpha_j-\beta_j-\gamma_j)
$$

and

$$
\Delta_j \equiv -\sum_{n=1}^{N} \lambda_{jn}((1+\beta_{jn})^{\alpha_j} - 1) - \sum_{m=1}^{M} \lambda_{jm}\left((1+\beta_{jm})^{1-\alpha_j-\beta_j-\gamma_j} - 1\right)
$$

If we impose the following parameter constraint originally suggested by Xie (1991) and extensively used in the literature since then,

$$
\phi_j = \beta_j \tag{9}
$$

then we can find the closed-form representation of the value function (that satisfies both the HJB equation and the transversality condition, or TVC) of the form

$$
V_j(K_j, A_j, H_j, S_j) = \mathbb{X}_j K_j^{1-\beta_j} + \mathbb{Y}_j A_j^{\alpha_j} H_j^{\gamma_j} S_j^{1-\alpha_j-\beta_j-\gamma_j} + \mathbb{Z}_j \tag{10}
$$

where

$$\mathbb{X}_j \equiv \frac{1}{1-\beta_j}\left(\frac{\beta_j}{\rho_j+(1-\beta_j)\delta_{jk}}\right)^{\beta_j}$$

$$\mathbb{Y}_j \equiv \frac{\vartheta_j^{1-\alpha_j-\beta_j-\gamma_j}}{b_j^{\gamma_j}}\left(\frac{\beta_j}{\rho_j}\right)^{\beta_j}\left(\frac{1-\gamma_j}{\rho_j-\alpha_j\mu_j-\gamma_j(b_j-\delta_{jh})+\nu_j(1-\alpha_j-\beta_j-\gamma_j)+\Theta_j+\Delta_j}\right)^{1-\gamma_j} \tag{11}$$

$$\mathbb{Z}_j \equiv -\frac{1}{\rho_j(1-\beta_j)}$$

The corresponding expressions for control variables are

$$C = \frac{\rho_j+(1-\beta_j)\delta_{jk}}{\beta_j}K \tag{12}$$

and

$$u = \frac{\rho_j-\alpha_j\mu_j-\gamma_j(b_j-\delta_{jh})+\nu_j(1-\alpha_j-\beta_j-\gamma_j)+\Theta_j+\Delta_j}{b_j(1-\gamma_j)} \tag{13}$$

Proof. See Appendix A[9].

2.3. Comments on Theorem

I in turn comment on the main points in Theorem 1.

2.3.1. Parameter Restriction and Value Function

The parameter restriction (9), originally proposed by Xie (1991), says that the risk aversion parameter equals the physical capital share of income. Whether it holds true in practice is still open debate, because the estimation of ϕ is a task of great difficulty. For example, on the one hand, Lucas (2003) claims that ϕ ranges from 1 (logarithmic utility) to 4, but on the other, Smith (2007) argues that ϕ should be smaller than 1. Despite this empirical controversy, the restriction (9) has been widely used by a number of authors in order to obtain the closed-form solution to their model. Xie (1991, 1994), Rebelo and Xie (1999), Smith (2007), Bucci et al (2011), Marsiglio and La Torre (2012a, 2012b), Hiraguchi (2013), and Tsuboi (2018), all use the restriction (9), and thus I follow them, as it allows us to inspect the underlying mechanism in the most transparent way.

At a first glance, imposing the parameter restriction may look problematic. But this is one of the major approaches in the literature on continuous-time methods under uncertainty. For example, Wälde (2011b, p.277) states that "For a much larger class of models - which

[9] The condition for $u \in (0,1)$ is lengthy and can easily be obtained by straightforward calculation.

are then standard models - closed-form solutions cannot be found...Economists then either go for numerical solutions...or they restrict the parameter set in a useful way. Useful means that with some parameter restriction, value functions can be found again and closed-form solutions are again possible."

One may wish to back down the parameter restriction (9) and thereby the closed-form representation of the value function (10), and instead resort to (say) the value function iteration or finite-difference method. Although they can be another approach, they too have some shortcomings. For instance, there is no guarantee that the value function would converge to the "true" one. Or, even if it does, without the analytical solution, it would be hard to see what is drawing one's findings. As we will see, the closed-form solution at our disposal is essential for perspicuously understanding how major macroeconomic variables interact with each other. As such, the real problem is not the parameter restriction itself: what really matters is to evaluate which "cost" - the cost of imposing the parameter restriction and the cost of numerical approximation - is higher. Here, following many studies cited above, I proceed on grounds that the former is lower.

Equation (10) is the closed-form representation of the value function that will be used in the welfare analysis below. We can see that physical capital and the product of technology, human capital, and exhaustible resources are separable. Note that, again, when $\sigma_{js} = \eta_j = 0$ and there is no Poisson uncertainty, the value function (10) completely coincides with that of Hiraguchi (2013, equation (29)). The non-separability here implies that endogenous growth comes from a fusion of technological progress, the accumulation of human capital, and use of exhaustible resources.

2.3.2. Control Variables

Equation (12) tells us that the consumption-capital ratio is constant. It seems a bit at odds that the optimal level of consumption depends only on K, not on the rest of three state variables. Moreover, it is irrelevant to uncertainty terms such as σ_{js} and λ_{jm}. However, Wälde (2011a) and Hiraguchi (2013) also observe this sort of property. Since it is found in the one-sector stochastic growth model of Smith (2007) as well, the optimal level of consumption appears to linearly depend on physical capital stock only. Tsuboi (2018) solves the puzzle that consumption is independent of shock terms by analytically solving the stochastic Uzawa-Lucas model with risky physical capital.

Equation (13) says that the time spent in working is constant as well, again consistent with Hiraguchi (2013). You can see that u involves key parameters relevant to stochastic processes. Here, the most important difference between this paper and Hiraguchi (2013) is that, the effect of diffusion coefficients σ_{ja} and σ_{js} on u is *indeterminate*. Specifically, u is always increasing in σ_{ja} in Hiraguchi (2013, p.137). However, in sharp contrast to the previous studies, because there are *two* diffusion processes that are *correlated*, the effect of one shock is dependent on the other, and can be indeterminate.

2.4. Growth and Welfare

The above observation has remarkable implications for the growth rate of human capital (and welfare), which is the quintessence of the Uzawa-Lucas model. In fact, one can show that, from the deterministic differential equation (1) and time spent in working (13), the growth rate of human capital $\mathcal{G}_j^h$ is given by

$$\mathcal{G}_j^h \equiv \frac{\dot{H}_j}{H_j} = \frac{b_j - \rho_j - \delta_{jh} + \alpha_j \mu_j - \nu_j(1 - \alpha_j - \beta_j - \gamma_j) + \Theta_j + \Delta_j}{1 - \gamma_j}$$

where $\dot{H}_j \equiv dH_j/dt$. First, note that, in the absence of technological progress ($\mu_j = 0$), depreciation ($\delta_{jh} = 0$), resource extraction ($\nu_j = 0$), and uncertainty terms, the sign of $\mathcal{G}_j^h$ depends exclusively on the relative size of the efficiency parameter of human capital accumulation b_j and the subjective discount rate of households ρ_j. As Kuwahara (2017) discusses in detail, this is the usual property of the deterministic Uzawa-Lucas model. It turns out that my model also has that property, as it should.

As the seminal paper of Barlevy (2004) shows, growth rates have close ties with welfare. In consequence, I discuss the growth and welfare implications *in parallel*. For instance, as we have the closed-form representation of the value function (10) and that of growth rate of human capital, one can analytically confirm that *what accelerates growth rate of human capital is absolutely welfare-improving*, and vice versa.

Second, since η_j is one of the most important parameters in this paper, we have to understand its impact on growth rate of human capital. One can show that

$$\frac{\partial \mathcal{G}_j^h}{\partial \eta_j} > 0$$

thus, higher correlation raises the growth rate of human capital in country j. To see why, notice that the proportion of time devoted to learning u_j is decreasing in η_j, as shown in (13). This means that higher η_j discourages people to work, or equivalently, encourages them to accumulate their new human capital. Therefore, since the accumulation of human capital is accelerated, the growth rate of human capital increases in response to higher correlation between two stochastic processes[10]. Since this is the important mechanism through which parameters affect growth rate of human capital and welfare (most of analyses below can be understood via this channel), it would be worth illustrating.

Figure 2 displays the relationship between the degree of correlation η_j and time allocation u_j (the left panel) and welfare V_j[11] (or equivalently, growth rate of human capital, the right panel)[12]. The left panel shows that, as I have just explained, the higher correlation lowers

[10]By the same token, one can immediately see that $\mathcal{G}_j^h$ is raised by the more efficient accumulation of human capital ($\partial \mathcal{G}_j^h / \partial b_j > 0$) and higher growth rate of technology ($\partial \mathcal{G}_j^h / \partial \mu_j > 0$), while it is reduced by the higher depreciation rate of human capital ($\partial \mathcal{G}_j^h / \partial \delta_{jh} < 0$) and higher extraction rate ($\partial \mathcal{G}_j^h / \partial \nu_j < 0$).

[11]Following Turnovsky (1997, 2000) and Tsuboi (2018), I use the value function $V_j(K_j, A_j, H_j, S_j)$ for welfare analyses.

[12]In the spirit of Mankiw et al (1992, p.432), I set $\alpha_j = \beta_j = \gamma_j = 0.03$. $b_j = 0.10$ is the value when Barro and Sara-i-Martin (2004) use in simulating the Uzawa-Lucas model. I choose $\mu_j = 0.02$ and $\delta_{jk} = \delta_{jh} = 0.03$ again following Mankiw et al (1992). $\nu_j = 0,01$, $\vartheta_j = 1$, and $\sigma_{js} = \sigma_{ja} = \lambda_{jn} = \lambda_{jm} = \beta_{jn} = \beta_{jm} = 0.01$ are set purely for the illustrative purpose. Finally, following Caballé and Santos (1993) and Moll (2014), I set $\rho = 0.05$. Note that, unlike Xu (2017), the purpose of this paper is not the quantitative assessment of the policy.

the proportion of time devoted to working, and increases that devoted to learning. As it accelerates the accumulation of human capital, its growth rate is enhanced. Moreover, since the value function V_j is the function of state variables, more H_j improves welfare (the right panel). From this illustration, you can visually see that time allocation is the key to understanding the implications for growth rate of human capital, and hence welfare.

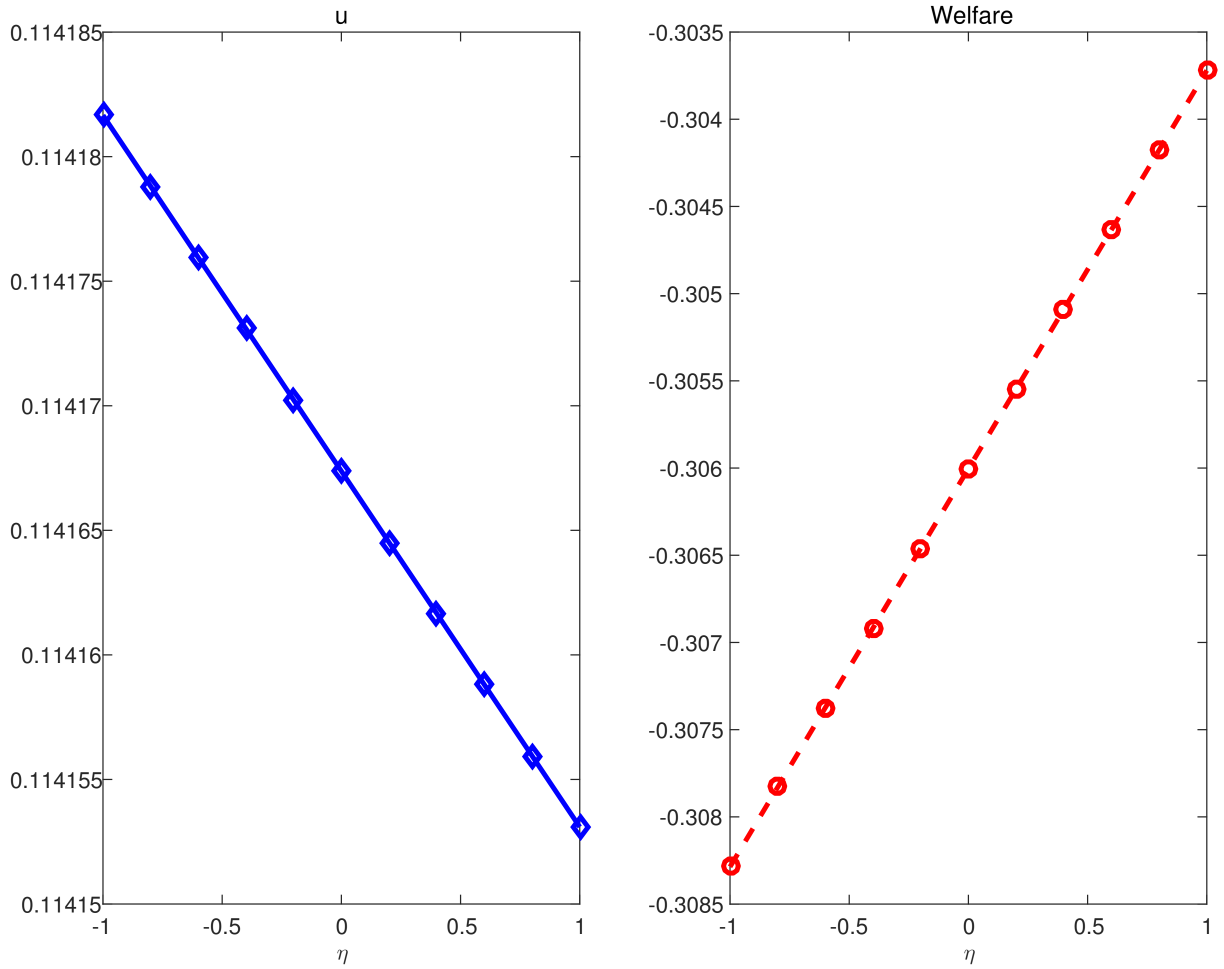

Figure 2: The relationship between correlation of two stochastic processes η_j and time allocation u_j (the left panel) and welfare V_j (the right panel). Parameters are $\alpha_j = \beta_j = \gamma_j = 0.03$, $b_j = 0.10$, $\mu_j = 0.02$, $\delta_{jk} = \delta_{jh} = 0.03$, $\nu_j = 0, 01$, $\vartheta_j = 1$, $\sigma_{js} = \sigma_{ja} = \lambda_{jn} = \lambda_{jm} = \beta_{jn} = \beta_{jm} = 0.01$, and $\rho_j = 0.05$. The higher correlation increases the time devoted to learning, hence accelerates the growth rate of human capital and improves welfare.

Third, what about the impact of resource shock σ_{js} on $\mathcal{G}_j^h$? Unlike parameters already discussed above, the sign is not determinate, for

$$\frac{\partial \mathcal{G}_j^h}{\partial \sigma_{js}} \begin{cases} > 0 & (\text{if } \sigma_{js} < \frac{\alpha_j \eta_j}{\alpha_j+\beta_j+\gamma_j}\sigma_{ja}) \\ < 0 & (\text{if } \sigma_{js} > \frac{\alpha_j \eta_j}{\alpha_j+\beta_j+\gamma_j}\sigma_{ja})) \end{cases} \tag{14}$$

In other words, the effect of higher resource uncertainty on the growth rate of human capital (and welfare) is ambiguous. To understand this point, see Figure 3. It displays the relationship between between the size of resource shocks σ_{js} and welfare V_j. Each line presented are indexed by the correlation coefficient η_j.

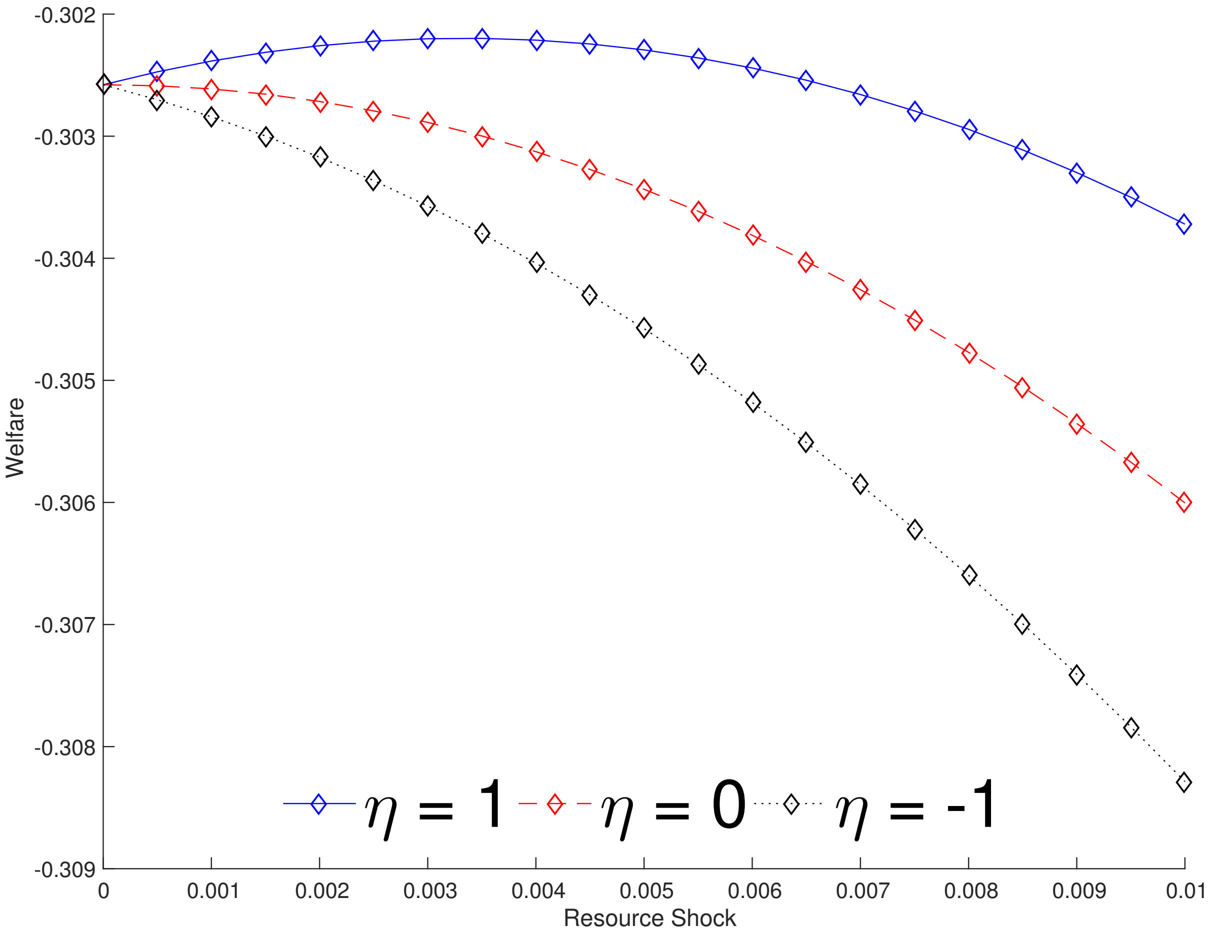

Figure 3: The relationship between the size of resource shocks σ_{js} and welfare V_j. Parameters are $\alpha_j = \beta_j = \gamma_j = 0.03$, $b_j = 0.10$, $\mu_j = 0.02$, $\delta_{jk} = \delta_{jh} = 0.03$, $\nu_j = 0, 01$, $\vartheta_j = 1$, $\sigma_{js} = \lambda_{jn} = \lambda_{jm} = \beta_{jn} = \beta_{jm} = 0.01$, and $\rho_j = 0.05$. When η_j is nonpositive, higher resource uncertainty, as usual, reduces welfare. However, when $\eta_j > 0$, higher resource uncertainty improves welfare, as long as the shock is small.

We begin with the case of no correlation $\eta_j = 0$. In this case, as represented by the dashed line, higher uncertainty reduces welfare. To see why, note that u_j is increasing in σ_{js}. This means that higher uncertainty discourages people to accumulate their new human capital, as there is uncertainty in human capital sector. As a result, people spend more time in the production sector, leading to human capital contraction. Then, since the stock of human capital decreases, the growth rate of human capital is decreased, and welfare is deteriorated. The case of negative correlation $\eta_j < 0$ can be interpreted in a similar way.

The interesting case would be when $\eta_j > 0$, that is, when two stochastic processes are positively correlated. In this case, we can see a *hump-shaped* relationship between welfare and the size of shocks. To understand this, remember that, as we saw above, the higher correlation strengthens the growth rate and enhances welfare, since it leads to more accumulation of human capital. Thus, in this case, there are two conflicting forces - the "accumulation" effect due to higher correlation and the "contraction" effect due to higher uncertainty. For a moderate degree of resource uncertainty, the former effect outweighs the latter, hence the net result is the accumulation of human capital, which raises its growth rate and improves welfare. However, beyond the threshold value at which the equality $\sigma_{js} = \frac{\alpha_j \eta_j}{\alpha_j+\beta_j+\gamma_j}\sigma_{ja}$ holds, the latter outweighs the former, resulting in the contraction of human capital. As such, the threshold value, *which is available thanks to the closed-form solution*, is the point where the relative "power" of two conflicting forces changes, thereby yielding a hump-shaped relationship between welfare and the size of shocks.

Although this point would be hard to swallow, another interpretation (which might be clearer and more intuitive) is to observe that, in response to higher resource uncertainty, households tend to spend more time in learning, thereby accelerating the accumulation of human capital. For a moderate degree of uncertainty, this positive effect of uncertainty dominates its standard negative impact due to risk aversion, thus leading to a net welfare gain. This is the reason why we see the positive relationship between uncertainty and welfare, as in real business cycle (RBC) models of Cho et al (2015) and Lester et al (2014), and the continuous-time stochastic AK model of Xu (2017). However, when shocks to resources are large enough, the usual negative effects outbalance the positive impact, resulting in a net welfare loss. As a consequence, there exists a hump-shaped relationship between uncertainty and welfare[13].

Moreover, since we have

$$\frac{\partial \mathcal{G}_j^h}{\partial \sigma_{ja}} \begin{cases} > 0 & \left(\text{if } \sigma_{ja} < \frac{(1-\alpha_j-\beta_j-\gamma_j)\eta_j}{1-\alpha_j}\sigma_{js}\right) \\ < 0 & \left(\text{if } \sigma_{ja} > \frac{(1-\alpha_j-\beta_j-\gamma_j)\eta_j}{1-\alpha_j}\sigma_{js}\right) \end{cases} \tag{15}$$

one can easily gauge that the impact of technology shocks on the growth rate of human capital and welfare is again ambiguous, and that we would see the same patterns described in Figure 3. As you can see in Figure 4, the underlying mechanism through which the hump-shaped pattern emerges is completely the same with the one for resource shocks. Therefore, I refrain from repeating the same explanation above.

Finally, what about the impact of jumps on the growth rate of human capital and welfare? To begin with, the larger size of both kinds of jump is *always* growth- and welfare-enhancing ($\partial \mathcal{G}_j^h/\partial \beta_{jn} > 0$ and $\partial \mathcal{G}_j^h/\partial \beta_{jm} > 0$ for all n and m) since they accelerate the accumulation of human capital. Intuitively, when the return from the investment in human capital suddenly gets higher due to resource or technology jumps, it will be more attractive

[13]You may think that this is possible because of the parameter restriction (9), which restricts $\phi_j < 1$, making people less risk-averse than "usual" ($\phi_j = 1$). It is not true, however. In fact, Lester et al (2014) confirm that, in some cases, higher uncertainty improves welfare for a very wide range of risk aversion parameter; from $\phi_j = 0.5$ to $\phi_j = 5.0$ (hence higher risk aversion).

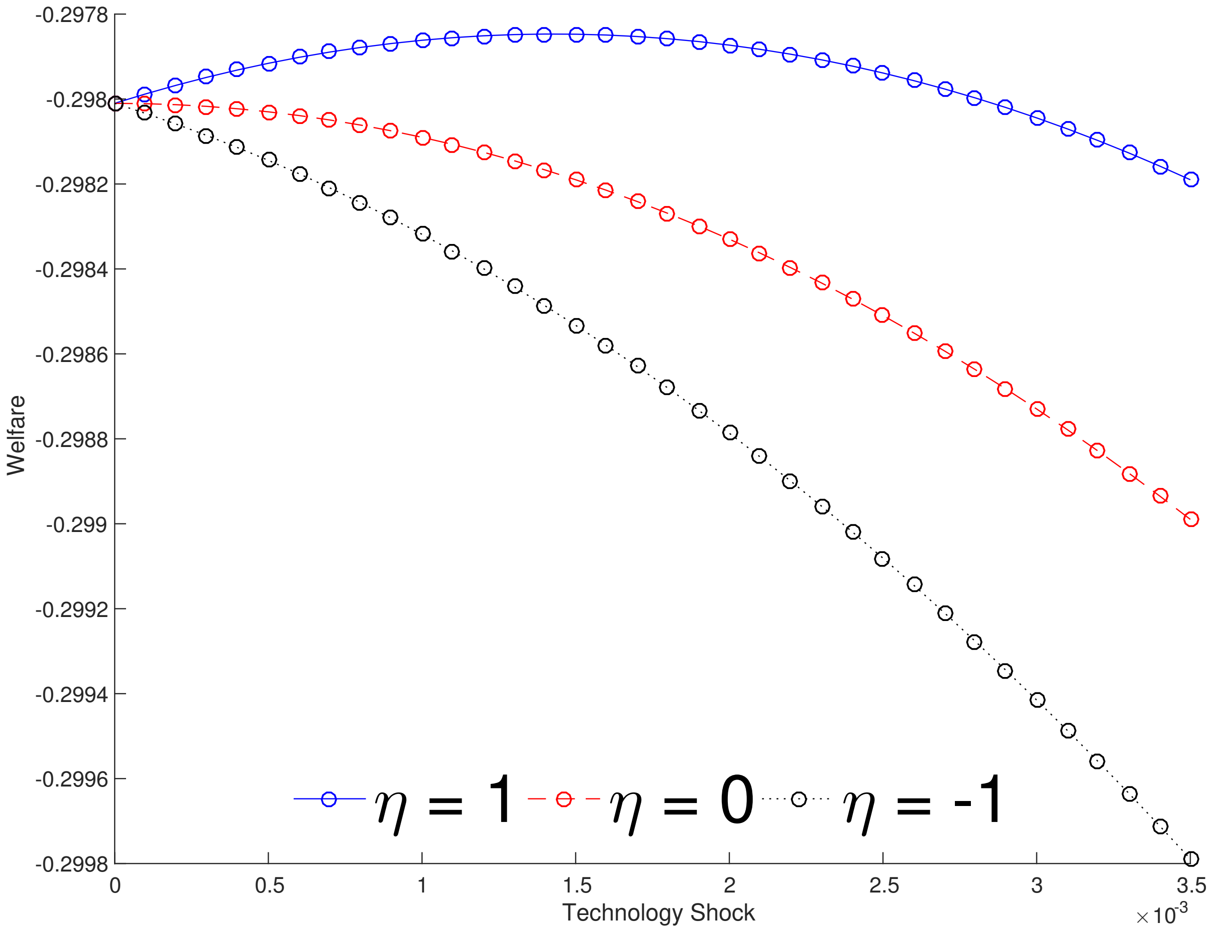

Figure 4: The relationship between the size of technology shocks σ_{ja} and welfare V_j. Parameters are $\alpha_j = \beta_j = \gamma_j = 0.03$, $b_j = 0.10$, $\mu_j = 0.02$, $\delta_{jk} = \delta_{jh} = 0.03$, $\nu_j = 0, 01$, $\vartheta_j = 1$, $\sigma_{js} = \lambda_{jn} = \lambda_{jm} = \beta_{jn} = \beta_{jm} = 0.01$, and $\rho = 0.05$. When η_j is nonpositive, larger technology shocks, as usual, deteriorate welfare. However, when $\eta_j > 0$, larger technology shocks improve welfare, as long as the shock is small.

for people to invest in human capital. As a consequence, the larger jumps promote human capital accumulation, and in turn, increase the growth rate of human capital, and thereby improving welfare.

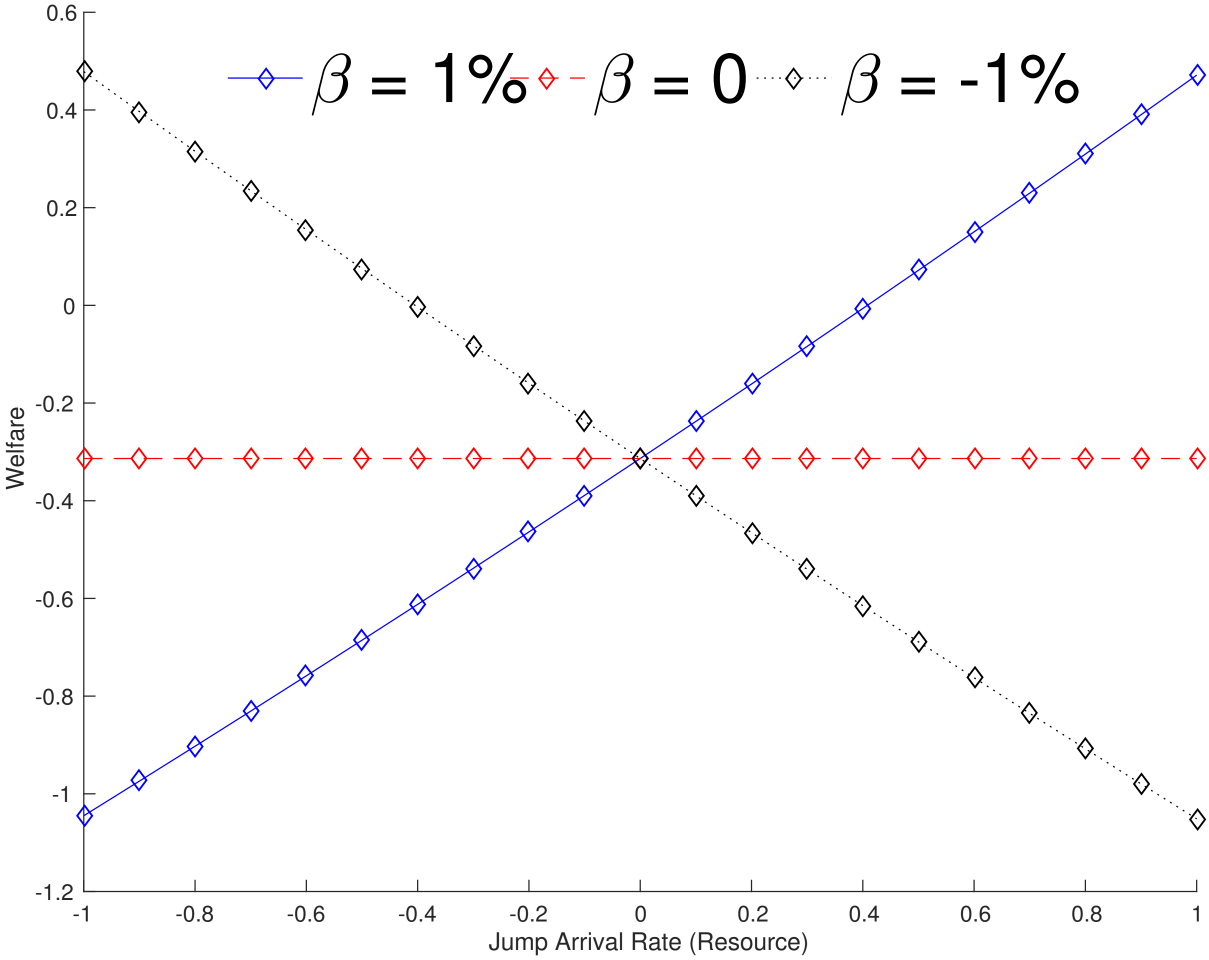

Figure 5: The relationship between the jump arrival rate of resources λ_{jm} and welfare V_j. Parameters are $\alpha_j = \beta_j = \gamma_j = 0.03$, $b_j = 0.10$, $\mu_j = 0.02$, $\delta_{jk} = \delta_{jh} = 0.03$, $\nu_j = 0,01$, $\vartheta_j = 1$, $\sigma_{js} = \lambda_{jn} = \lambda_{jm} = \beta_{jn} = \beta_{jm} = 0.01$, and $\rho = 0.05$. When a jump of size β_{jm} is positive, welfare is improved with the higher arrival rate, and vice versa.

On the other hand, the implications of arrival rates λ_{jn} and λ_{jm} would be more appealing, since for all m, we have

$$\frac{\partial \mathcal{G}_j^h}{\partial \lambda_{jm}} \begin{cases} > 0 & (\text{if } \beta_{jm} > 0) \\ < 0 & (\text{if } \beta_{jm} \in (-1, 0)) \end{cases} \tag{16}$$

that is, effects of the arrival rate of exhaustible resources are indeterminate. The sign depends whether the size of jump is greater or less than zero. To get some intuition, see

Figure 5. It shows the relationship between welfare V_j and the arrival rate of exhaustible resources λ_{jm}. Each line presented is indexed by the jump size $\beta_{jm} = 1\%, 0\%$, and -1%. The $\beta_{jm} = 0\%$ line is plotted for the benchmark with no jumps. For this illustration, I assume that there is only one Poisson jump process ($M = 1$).

You can see that, when $\beta_{jm} = 1\%$, the higher arrival rate improves welfare. As in the above case, the higher arrival rate causes people to spend more time in learning, because when the return from investment in human capital is likely to be positive (due to the positive jump of resources) and hence attractive, it would be desirable for the "event" to happen more frequently. Therefore, it leads to the accumulation of human capital, resulting in higher growth rate of human capital and welfare enhancement. Unlike η_j, as there are not conflicting forces, this mechanism would be clearer to understand. The case of $\beta_{jm} = -1\%$ and of technology jump (β_{jn} and λ_{jn}) can be similarly explained, and it is illustrated in Figure 6.

The findings of this section can be summarized as follows:

Proposition 1. *One parameter restriction commonly used by many previous studies makes it possible to find the closed-form solution to the stochastic Uzawa-Lucas model in which both technological progress and the depletion of exhaustible resources are driven by the mixture of correlated Brownian motion and independent many Poisson jump processes. The higher correlation between two stochastic processes and bigger jumps always increases growth rate of human capital and improves welfare. The effect of arrival rates on growth rate of human capital and welfare depends on whether jump size is positive or negative. When two stochastic processes are positively correlated, there exists a hump-shaped relationship between resource/technology uncertainty and welfare, as long as the size of shocks is small.*

3. Concluding Remarks

Will shortages of natural resources constrain economic growth? As the amount of natural resources on earth is fixed, the answer seems obviously yes. However, there are two reasons why this may not necessarily true. The first argument is that resource scarcity can be undone by the resource-saving technological progress. The second is that, at the country level, countries can import necessary resources in production from other countries. This paper analyzes the situation in which these two excuses are less valid.

For technology, we know that not all technological progress is resource-saving, and its arrival is not predictable (Dasgupta and Heal, 1974). For the import argument, at the global level, we also know that the world as a whole cannot make up for a shortage of natural resources by importing. To handle these, I construct the open, stochastic Uzawa-Lucas endogenous growth model in continuous time. I then analytically show that the answer is sensitive to the interaction between technology shocks and resource shocks, and find that, in some cases, higher uncertainty is not necessarily detrimental to welfare (Cho et al., 2015, Lester et al., 2014, and Xu, 2017).

To summarize for the purposes of policy implications, technological progress is welcome, as it is the ultimate engine of growth (Jones, 2016) and probably improves welfare. However,

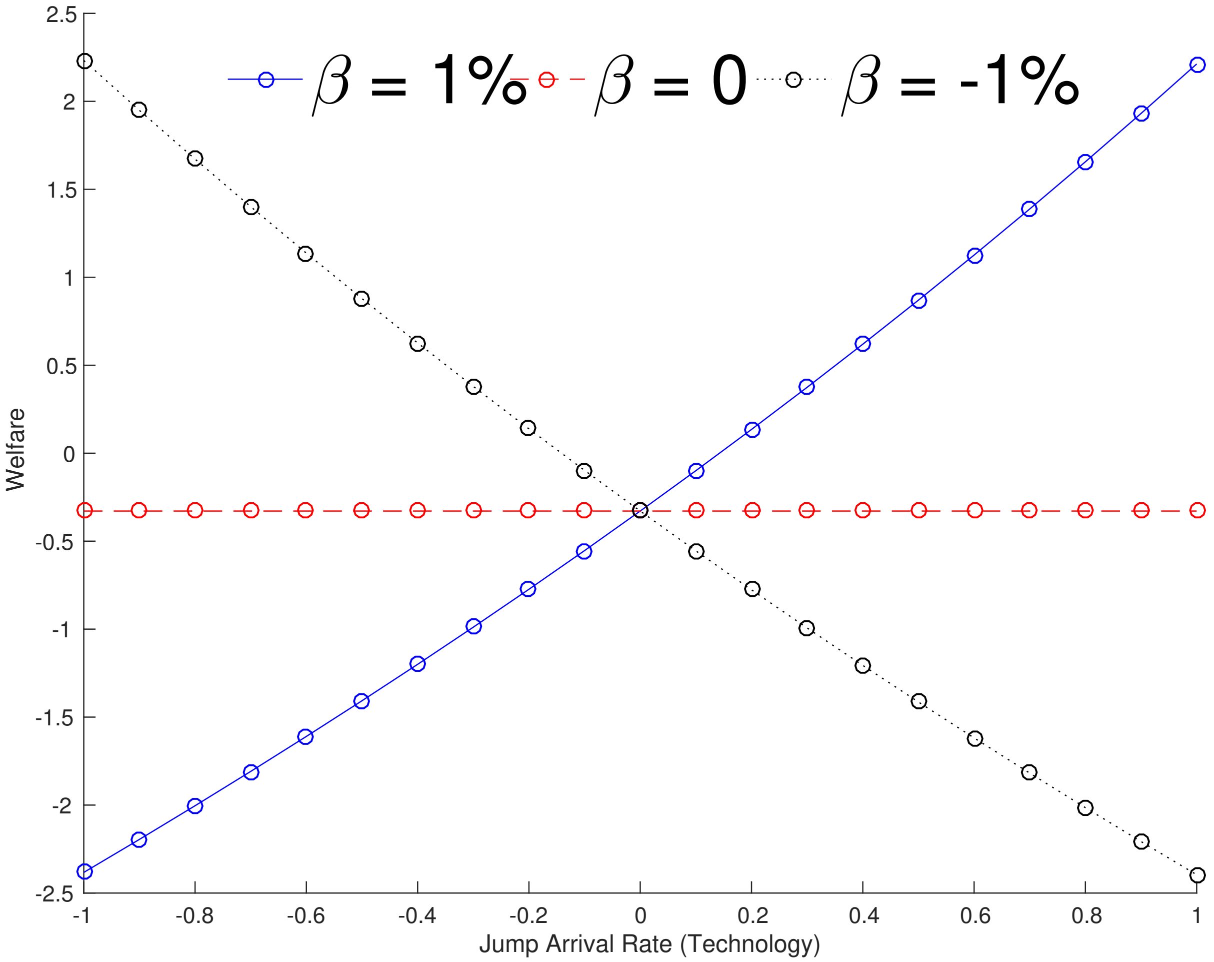

Figure 6: The relationship between the arrival rate of technology λ_{jn} and welfare V_j. Parameters are $\alpha_j = \beta_j = \gamma_j = 0.03$, $b_j = 0.10$, $\mu_j = 0.02$, $\delta_{jk} = \delta_{jh} = 0.03$, $\nu_j = 0,01$, $\vartheta_j = 1$, $\sigma_{js} = \lambda_{jn} = \lambda_{jm} = \beta_{jn} = \beta_{jm} = 0.01$, and $\rho = 0.05$. When a jump of size (β_{jn}) is positive, the higher arrival rate improves welfare, and vice versa.

from the resource scarcity viewpoint, it is not *always* welcome, as not all types of technological progress are resource-saving. Moreover, we cannot precisely know whether the arrival of resource-saving technological progress will repeat in the future. Therefore, the answer to the above question is not definitive; as the global level, if we can promote the resource-saving technological progress and make it arrive more often, the resource scarcity may be undone. However, if we fail to promote the resource-saving technological progress, or if its arrival process fails to repeat in the future, resource scarcity may pose a serious threat to growth, and to the welfare of humanity.

This paper lends itself to several extensions. First, in order to preserve the closed-form solution, I throughout assume that resources are not exchanged among countries. This assumption, however, is utterly unrealistic. It would be desirable to consider the setting where countries exchange natural resources, while keeping the explicit solution available. Although I believe that the implications of the present model remain unchanged by that consideration, they may be altered in that situation. Second, the natural resource analyzed in this paper is *exhaustible* resources. However, as not all resources are exhaustible, considering the case of *renewable* resources may enrich the analysis and generate the new insights into the resource-growth, and resource-welfare, nexus.

Appendix A. Guide to Analytical Solutions

This appendix briefly describes how to find the closed-form representation of the value function (10) in Theorem 1. For this purpose, postulate the tentative value function of the form

$$V_j(K_j, A_j, H_j, S_j) = \mathbb{X}_j K_j^{\theta_1} + \mathbb{Y}_j H_j^{\theta_2} A_j^{\theta_3} S_j^{\theta_4} + \mathbb{Z}_j$$

where $\mathbb{X}_j$, $\mathbb{Y}_j$, $\mathbb{Z}_j$, θ_1, θ_2, θ_3, and θ_4 are all unknown constants to be determined. The relevant partials are $V_{jK} = \mathbb{X}_j\theta_1 K_j^{\theta_1-1}$, $V_{jKK} = \mathbb{X}_j\theta_1(\theta_1-1)K^{\theta_1-2}$, $V_{jH} = \mathbb{Y}_j\theta_2 H_j^{\theta_2-1}A_j^{\theta_3}S_j^{\theta_4}$, $V_{jA} = \mathbb{Y}_j\theta_3 H_j^{\theta_2}A_j^{\theta_3-1}S_j^{\theta_4}$, $V_{jAA} = \mathbb{Y}_j\theta_3(\theta_3-1)H_j^{\theta_2}A_j^{\theta_3-2}S_j^{\theta_4}$, $V_{jS} = \mathbb{Y}_j\theta_4 H_j^{\theta_2}A_j^{\theta_3}S_j^{\theta_4-1}$, $V_{jSS} = \mathbb{Y}_j\theta_4(\theta_4-1)H_j^{\theta_2}A_j^{\theta_3}S_j^{\theta_4-2}$, and $V_{jSA} = \mathbb{Y}_j\theta_3\theta_4 H_j^{\theta_2}A_j^{\theta_3-1}S_j^{\theta_4-1}$.

To obtain the explicit expression, substitute these partials into the maximized HJB equation (8). Then, set $\theta_1 = 1-\beta_j$, $\theta_2 = \gamma_j$, $\theta_3 = \alpha_j$, and $\theta_4 = 1-\alpha_j-\beta_j-\gamma_j$. Finally, imposing the parameter restriction (9), you can find the explicit expressions for $\mathbb{X}_j$, $\mathbb{Y}_j$, and $\mathbb{Z}_j$, and consequently, those for control variables C_j and u_j and for the value function $V(K_j, A_j, H_j, S_j)$ in Theorem 1.

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