The Elasticity of Intertemporal Substitution and the Wealth Inequality in a Global Economy*

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Abstract

This paper examines the role of preferences for the distribution of wealth in a global economy. In the dynamic Heckscher-Ohlin model, we firstly find that the steady-state equilibrium is uniquely determined regardless of the homothetic and non-homothetic utility function. Secondly, we show that the distribution of wealth in world economy shrinks or expand. In particular, in the case of homothetic utility function, the dispersion of wealth shrinks (expands) if the speed of convergence is fast (or, slow). Alternatively, in the case of non-homothetic utility function, this result may not be held. Thirdly, when the identical utility function is homothetic, the characterization of wealth inequality in domestic economy is qualitatively the same with that in world economy. Alternatively, if the utility function is non-homothetic, this result may not be held.

Keywords: Homothetic and non-homothetic utility function; Wealth distribution; Two countries

JEL Classification Code: F43; O41

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1 Introduction

It has been well known that the Heckscher-Ohlin propositions can be established in a dynamic $2 \times 2 \times 2$ model of the word economy in which the representative household in each country accumulates capital. As well as in the static Heckscher-Ohlin setting, the key assumptions in the dynamic Hechscher-Ohlin model are: (i) both country have the identical neoclassical technologies; (ii) the households in both country have identical, homothetic utility functions and the same time discount rate; (iii) both consumption and investment goods are freely traded, whereas capital and labor cannot move across the border and; (iv) the only difference between the countries is the initial stock of capital held by the households. Given those conditions, the dynamic behavior of the world economy is independent of capital distribution between the two countries. Moreover, the world economy may converges to a unique steady state, and the long-run distribution of capital between the two countries depends on its initial distribution. Under the standard neoclassical assumptions, the converging path towards the steady state is unique and monotonic, so that the initial distribution of capital determines the long-run distribution. This means that the country whose initial capital is larger than the other country may keep the comparative advantage in producing goods that use relatively capital intensive technology. Namely, the long-run patterns of trade are determined by the initial distribution of capital between the two countries, which represents a dynamic version of the Heckscher-Ohlin theorem.

The purpose of this paper is to reconsider the dynamic Heckscher-Ohlin model in a more general setting than the baseline, dynamic two-country model. Our main departure from the standard setting is that households in each country are heterogeneous and their preferences may not be homothetic. As to the heterogeneity of households, it is assumed that the each household's initial stock of capital is not the same. If the utility function is homothetic, such kind of heterogeneity will not affect the dynamic behavior of each country as well as the world economy dynamics. However, if households in each country have different forms of utility functions, the distribution of capital among the households affects not only the behavior of each country but also the world economy dynamics. In this situation, the initial distribution of capital among the households affects the long-run distribution of capital between the countries so that it may determine the long-term patterns of trade. The central concern of this paper is the relation between wealth distribution among households within each country and the long-run behavior of the two country world.

In this paper we first consider the general case where preferences of households in the world economy are heterogeneous, while the production technologies are symmetric between the two countries. We demonstrate that if preferences and technologies satisfy the neoclassical properties, the world economy has a unique steady state and it monotonically converges to the steady state equilibrium. Then we examine the case in which preferences are heterogeneous but each household has a homothetic utility function. In this setting, we examine patterns of wealth distribution among the households in each country as well as that between the countries. Finally, we examine the model in which all the households in the world economy have identical, non homothetic utility functions. As far as the preference structure concerned, this model has a least difference from the standard dynamic Heckscher-Ohlin model. We demonstrate that even in this environment, the distribution of wealth within a country may have decisive impacts on the international distribution of wealth in the long run.

Related Literature

Our study is closely connected to two groups of literature: the dynamic Heckscher-Ohlin modesl and the neoclassical growth models with heterogeneous households. An early study on the dynamic versions of Heckscher-Ohlin propositions is presented by Stiglitz (1970). Chen (1992) studies a $2\times2\times2$ model of neoclassical growth and drives the long-run Heckscher-Ohlin theorem. Recent studies on this subject include Cremers (1997), Nishimura and Shimomura (2002), Hu et al (2009), Bond et al. (2011) and Caliendo (2011). Among others, Bond et al. (2011) analyze a model with non-homothetic preferences[1]. Their discussion, however, assumes that there is a representative household in each country, so that they do not argue wealth distribution among the households.

As to the neoclassical growth models with heterogeneous households, a number of authors examine dynamic wealth distribution in closed economies. The early contributions to this literature are Becker (1980) and Stiglitz (1969). Further studies are given by Bourguignon (1980), Chatterjee (1984), Foellmi (2011), Kraus and Serve (2000) and Sorger (2002). In addition, Mino and Nakamoto (2012 and 2015) examine the wealth distribution and equilibrium

[1]See also Hunter (1991) for the role of non-homotheic preference in trade theory.

dynamics of neoclassical growth models with consumption externalities. While the literature cited above assumes preference heterogeneity, several authors such as Garcia-Penerosa and Turnovsky (2006 and 2009) and Caselli and Ventura (2000) study the models with homothetic preferences in which distribution of wealth among households does not affect the aggregate behavior of the economies[2]

Our study integrates those two classes of literatures into a single framework. We intend to show that such an integration may provide us with a useful analytical framework for investigating distribution, trade and growth of the world economy in a tractable manner.

2 Analytical Framework

We consider a world economy with two countries, home and foreign. In each country there is a continuum of households with a unit measure where households are assumed to be heterogeneous in the sense that they have different stock of wealth at the outset. As for the households' preferences, we assume that the households in both countries have an identical rate of time preference and an identical form of utility function, $u_j = u_j(C_j)$ in home country and $u_j^* = u_j(C_j^*)$ in foreign country. Since the production technologies are symmetric and the utility functions in each country are given by an identical form, we focus on the set-up in home country unless the confusion does not arise.

2.1 Production

The production side in our economy consists of two sectors: the sector i in each country produces investment goods and the sector c produces consumption goods. The production structure in our model is standard in the sense that the standard constant-return-to-scale neoclassical production technologies prevail in these two sectors. Therefore, supposing that the production function in sector $(h = i, c)$ is given by $Y_h = F_h(K_h, L_h)$ where Y_h is output, K_h is capital and L_h is labor in each sector $h = i, c$, the real rent, r, and real wage rate, w,

[2] Bertola et al. (2006) preset a useful survey of income distribution in dynamic macroeconomic models from a very broad perspective. Chapter 4 of their book includes a detailed exposition of wealth and income distribution of neoclassical growth models with heterogeneous households.

in competitive factor and product markets are determined by:

$$r = f_i'(k_i) = pf_c'(k_c), \tag{1a}$$

$$w = f_i(k_i) - k_i f_i'(k_i) = p\left\{f_c(k_c) - k_c f_c'(k_c)\right\}, \tag{1b}$$

where $k_x = K_x/L_x$ $(x = i, c)$ and p denotes the price of consumption good in terms of the investment goods. From these equations $(1a)$ and $(1b)$, the optimal factor intensity is given by the function of relative price:

$$k_i = k_i(p), \quad k_c = k_c(p), \tag{2}$$

For simplicity, we focus on the case where the investment good sector employs a more capital-intensive technology than the consumption good sector.

Assumption 1. *The invesment good sector uses a more capial intensive technology that the consumption good sector: $k_i(p) > k_c(p)$ for all feasible levels of p.*

From Assumption 1, it holds that

$$k_x = k_x'(p) < 0. \;\; x = i, \; c. \tag{3}$$

where

$$\frac{\partial k_x}{\partial p} = \frac{1}{p\left(k_i\left(p\right) - k_c\left(p\right)\right)} \frac{f_x'(k_x(p))^2}{f_x''(k_x(p))f_h(k_x(p))}$$

We suppose that the production factors freely shift between the sectors, but they cannot move between the countries. Denoting by K the level of aggregate capital in the home country, we give the full employment conditions for capital and labor respectively:

$$L_i + L_c = 1, \quad K_i + K_c = K, \tag{4}$$

where we assume that the labor supply is assumed to be constant and normalized to the unity. Using (2) and (4), we can lead to the following:

$$K = L_i k_i(p) + (1 - L_i)k_c(p), \tag{5}$$

and furthermore, the equation (5) can be rewritten as

$$L_i = \frac{K - k_c(p)}{k_i(p) - k_c(p)}, \tag{6}$$

where we suppose that $L_i \in (0,1)$, so that the two countries produce both consumption and investment goods. Consequently, the labor allocation to the investment sector is given by:

$$L_i = L_i(K,p) \equiv L(K,p). \tag{7}$$

Taking account of Assumption 1, we obtain each differential as follows:

$$\frac{\partial L}{\partial K}(\equiv L_K) = \frac{1}{k_i(p)-k_c(p)}(>0), \tag{8}$$

$$\frac{\partial L}{\partial p}(\equiv L_p) = \frac{k_c'(p)(K-k_i(p))}{(k_i(p)-k_c(p))^2} + \frac{k_i'(p)(k_c(p)-K)}{(k_i(p)-k_c(p))^2}(>0). \tag{9}$$

Making use of (7), each output function of investment and consumption goods is given by:

$$y^i(K,p) = L(K,p)f_i(k_i(p)), \quad y^c(K,p) = (1-L(K,p))f_c(k_c(p)), \tag{10}$$

where under the assumption that the investment good is capita-intensive, we show that

$$\frac{\partial y^i}{\partial K}(\equiv y^i_K) = L_K(K,p)f_i(p)(>0),$$

$$\frac{\partial y^c}{\partial K}(\equiv y^c_K) = -L_K(K,p)f_c(p)(<0),$$

$$\frac{\partial y^i}{\partial p}(\equiv y^i_p) = \frac{k_c'(p)(K-k_i(p))f_i(k_i(p))}{(k_i(p)-k_c(p))^2} + \frac{k_i'(p)(k_c(p)-K)f_i(k_i(p))}{(k_i(p)-k_c(p))^2}\left(1-\frac{f_i'(k_i(p))(k_i(p)-k_c(p))}{f_i(k_i(p))}\right)(>0),$$

$$\frac{\partial y^c}{\partial p}(\equiv y^c_p) = \frac{k_i'(p)(K-k_c(p))f_c(k_c(p))}{(k_i(p)-k_c(p))^2} + \frac{f_c(k_c(p))(k_i(p)-K)k_c'(p)}{(k_i(p)-k_c(p))^2}\left(1+\frac{f_c'(k_c(p))(k_i(p)-k_c(p))}{f_c(k_c(p))}\right)(<0).$$

2.2 Households

There is a contuuum households in each country with unit measure. The objective functional of household j in home country is given by

$$U^j = \int_0^\infty u_j(C_j)e^{-\rho t}dt, \quad \rho > 0, \quad j \in [0,1], \tag{11}$$

where ρ is the constant rate of time preference and common among the households and C_j is the level of household j's consumption. The flow budget constraint is:

$$\dot{k}_j = (r(p)-\delta)k_j + w(p) - pC_j. \tag{12}$$

where δ is the rate of depreciation.

Denoting the (private) utility price of capital by q_j, maximizing (11) subject to (12) yields the necessary conditions for an optimum:

$$u_j'(C_j) = q_j p, \tag{13a}$$

$$-\frac{\dot{q}_j}{q_j} = r(p) - \delta - \rho, \tag{13b}$$

together with the transversality condition $\lim_{t\to\infty} e^{-\rho t} q_j k_j = 0$. From (13$b$), it can be easily seen that $\frac{\dot{q}_j}{q_j} = \frac{\dot{q}_n}{q_n}$, which means that the ratio of marginal utility between these households j and n is constant over time.That is, it holds that $\frac{u_j'(C_j)}{u_n'(C_n)}$ = constant, and therefore, our model needs to specify the trajectory starting from a specific set of initial conditions which consist of the initial levels of capital holdings and the international difference of preferences.

Finally, summing up (12) among all agents in home country, we can see the dynamic equation of aggregate capital in home country:

$$\dot{K} = (r(p) - \delta)K + w(p) - pC, \tag{14a}$$

where $K = \int_0^1 K_j dj$ and $C = \int_0^1 C_j dj$. Furthermore, from $K_w = K + K^*$ and (14a), the capital accumulation equation in world economy is:

$$\dot{K}_w = (r - \delta)K_w + 2w - pC_w. \tag{14b}$$

2.3 World Market Equilibrium and the Capital Accumulation Equation

Let us consider the world market equilibrium. Because home and foreign countries produce both goods, all the firms in world economy face an identical value of relative price, p. Furthermore, the assumption of symmetric production structures in these countries yields an identical capital intensity, so that $k_h(p) = k_h^*(p)$ $(h = i, c)$ for all time.

Now, assuming that investment and consumption goods freely cross the borders under the free trade, the world market equilibrium conditions for both goods are given by:

$$y^c(K, p) + y^c(K^*, p) = C + C^*, \tag{15a}$$

$$y^i(K, p) + y^i(K^*, p) = \dot{K} + \dot{K}^* + \delta(K + K^*), \tag{15b}$$

where the levels of aggregate consumption in each country are given by $C = \int_0^1 C_j dj$ and $C^* = \int_0^1 C_j^* dj$. Defining the levels of world consumption and capital by $C_w = C + C^*$ and

$K_w = K + K^*$, from (15a), and making use of $k_x(p) = k_x^*(p)$ $(x = i, c)$, the world market condition for the consumption goods can be rewritten as:

$$C_w = \frac{2k_i(p) - K_w}{k_i(p) - k_c(p)} f_c(k_c(p)). \tag{16}$$

Using (16) yields the relative price which depends on the levels of world consumption and capital as follows:

$$p = p(K_w, C_w), \tag{17}$$

where

$$\frac{\partial p}{\partial K_w} = \frac{p}{Z_{hf}(2k_i(p) - K)}(< 0), \tag{18a}$$

$$\frac{\partial p}{\partial C_w} = \frac{p}{Z_{hf} C_w}(< 0), \tag{18b}$$

$$Z_{hf} = \frac{f_c'(k_c) p k_c'(p)}{f_c(k_c)} + \frac{p k_i'(p)(K_w - 2k_c(p))}{(2k_i(p) - K_w)(k_i(p) - k_c(p))} + \frac{k_c'(p) p}{k_i(p) - k_c(p)}(< 0). \tag{18c}$$

Since it holds that $k_x(p) = k_x^*(p)$ $(x = i, c)$, both countries face the same value of Z_{hf}.[3]

Finally, making use of (15b), we derive the world-capital accumulation equation. By replacing $y^i(K, p) + y^i(K, p)$ in (15b) by $L(K, p) f_i(k_i(p)) + L^*(K^*, p) f_i(k_i(p))$ and substituting (17) into (15b), we can modify the world market condition for the investment goods:

$$G(K_w, C_w) = \dot{K}_w + \delta K_w. \tag{19}$$

The function $G(K_w, C_w)$ is defined by:

$$G(K_w, C_w) \equiv \frac{K_w - 2k_c(p(K_w, C_w))}{k_i(p(K_w, C_w)) - k_c(p(K_w, C_w))} f_i(k_i(p(K_w, C_w))) \tag{20}$$

where

$$G_{K_w} = \frac{f_i(k_i)}{k_i - k_c} + \frac{f_i(k_i)\frac{\partial p}{\partial K_w}}{(k_i - k_c)^2}\left[k_c'(p)(K_w - 2k_i) + k_i'(p)(K_w - 2k_c)\left(\frac{(k_i - k_c) f_i'(k_i)}{f_i(k_i)} - 1\right)\right], \tag{21a}$$

$$G_{C_w} = \frac{f_i(k_i)\frac{\partial p}{\partial C_w}}{(k_i - k_c)^2}\left[k_c'(p)(K_w - 2k_i) + k_i'(p)(K_w - 2k_c)\left(\frac{(k_i - k_c) f_i'(k_i)}{f_i(k_i)} - 1\right)\right](< 0). \tag{21b}$$

[3] The variables with the subscript hf are common in both countries.

2.4 The Aggregate Consumption Path

To derive the dynamic equation of aggregate consumption, from (16) we lead to the dynamic equation of relative price:

$$\frac{\dot{p}}{p} = \frac{1}{Z_{hf}} \left(\frac{\dot{C}_w}{C_w} + \frac{K_w}{2k_i(p) - K_w} \frac{\dot{K}_w}{K_w} \right). \tag{22}$$

Turning back to the household optimization conditions in (13a) and (13b), we obtain:

$$\dot{C}_j = \omega_j \left(r(p) - \delta - \rho - \frac{\dot{p}}{p} \right), \tag{23a}$$

where we use $\omega_j = -u'_j(C_j)/u''_j(C_j)(> 0)$.

In particular, if the utility function is homothetic, the elasticity of intertemporal substitution ω_j/C_j is given by parameters alone. Therefore, an identity of utility function brings about an identity of elasticities of intertemporal substitution among agents, which implies that due to the identity of ω_j/C_j, the growth rate of private consumption, $\dot{C}_j/C_j$, is the same among agents. For instance, if we make use of the well-known constant-relative-risk-aversion type of utility function $u_j(C_j) = \frac{C_j^{1-\gamma}}{1-\gamma}$, the dynamic equation of private consumption, (23a) is given by

$$\frac{\dot{C}_j}{C_j} = \frac{r(p) - \delta - \rho - \frac{\dot{p}}{p}}{\gamma}, \tag{23b}$$

which can be seen that the right-hand side of (23b) is identical among agents. Alternatively, when the utility function is non-homothetic, the elasticity of intertemporal substitution is not only expressed by parameters, which means that the initially given dispersion of wealth is related to the difference of growth rate of private consumption among agents.

Summing up the equation (23a) of all households in home country yields:

$$\dot{C} = \Omega \left(r(p) - \delta - \rho - \frac{\dot{p}}{p} \right), \tag{23c}$$

where Ω is the sum of absolute risk aversion:

$$\Omega = \int_0^1 \omega_j dj (> 0). \tag{23d}$$

Similarly, the dynamic equation of aggregate consumption in the foreign country is given by:

$$\dot{C}^* = \Omega^* \left(r(p) - \delta - \rho - \frac{\dot{p}}{p} \right), \tag{24a}$$

$$\Omega^* = \int_0^1 \omega_j^* dj (> 0). \tag{24b}$$

where $\omega_j^* = -u_j^{*\prime}(C_j^*)/u_j^{*\prime\prime}(C_j^*)(> 0)$.

Finally, from (23c) and (24a) we can derive the following:

$$\dot{C}_w = (\Omega + \Omega^*) \left(r(p) - \delta - \rho - \frac{\dot{p}}{p} \right), \tag{25}$$

and substituting (22) into (25) yields the dynamic equation of aggregate consumption in total economy:

$$\dot{C}_w = \frac{Z_{hf} C_w (\Omega + \Omega^*)}{Z_{hf} C_w + \Omega + \Omega^*} \left(r(p) - \delta - \rho - \frac{G(K_w, C_w) - \delta K_w}{Z_{hf}(2k_i(p) - K_w)} \right), \tag{26}$$

where the relative price depends on the aggregate capital and consumption in world economy, $p = p(K_w, C_w)$ in (17).

Finally, as for the growth paths of personal consumption and total consumption in home country, substituting (22) into (23a) and (23c), and moreover, substituting (19) and (26) into these equations yields:

$$\frac{\dot{C}}{C} = \frac{Z_{hf} C_w \Omega / C}{Z_{hf} C_w + \Omega + \Omega^*} \left(r(p) - \delta - \rho - \frac{G(K_w, C_w) - \delta K_w}{Z_{hf}(2k_i(p) - K_w)} \right), \tag{27a}$$

$$\frac{\dot{C}_j}{C_j} = \frac{Z_{hf} C_w \omega_j / C_j}{Z_{hf} C_w + \Omega + \Omega^*} \left(r(p) - \delta - \rho - \frac{G(K_w, C_w) - \delta K_w}{Z_{hf}(2k_i(p) - K_w)} \right), \tag{27b}$$

where $p = p(K_w, C_w)$ in (17).

2.5 Steady State of the World Economy

We shall prove that the stationary states of the world economy exist where the steady-state values of each variable is expressed by the upper bar. The equations $\dot{K}_w = 0$ in (19) and $\dot{C}_w = 0$ in (26) are given by:

$$\delta \bar{K}_w = G(\bar{K}_w, \bar{C}_w), \tag{28a}$$

$$r(\bar{p}) = \delta + \rho, \tag{28b}$$

where the steady-state value of relative price is characterized by (17):

Proposition 1 *The steady-state equilibrium of world economy is uniquely determined. Furthermore, the unique steady state has the saddle-path stability if the following condition is satisfied:*

$$\bar{Z}_{hf}\bar{C}_w + \bar{\Omega} + \bar{\Omega}^* < 0. \tag{29}$$

Proof. See Appendix A. ■

It is to be noted that while the steady-state levels of aggregate consumption $\bar{C}_w$ and capital $\bar{K}_w$ in world economy and the relative price $\bar{p}$ are uniquely determined irrespective of the initial conditions such as the preference parameters and the initial levels of capital stock, the implicit prices of capital, $\bar{q}_i$ and $\bar{q}_i^*$, and the determination of steady-state levels of individual capital is needed to specify the initial conditions. Furthermore, the condition (29) is intuitively explained as follows. Making use of (29), the dynamic equation of aggregate consumption in world economy (26) has a positive sign of elasticities of intertemporal substitution, $\frac{Z_{hf}C_w(\Omega+\Omega^*)}{Z_{hf}C_w+\Omega+\Omega^*}(>0)$. As a result, the dynamic behavior of the aggregate consumption in world economy is intuitively reasonable in the sense that when the rate of return to capital is large, the growth rate of consumption in world economy has a positive sign, so the aggregate capital increases towards the steady state.

Supposing that the uniquely determined stable root of world economy is $\lambda(<0)$, on the saddle path of whole economy the relationship between consumption and capital in world economy is given by

$$C_w - \bar{C}_w = -\frac{G_{K_w} - \delta - \lambda}{G_{C_w}}(K_w - \bar{K}_w). \tag{30}$$

Making use of (30), the approximated behavior of aggregate capital in world economy is given by:

$$\dot{K}_w = \lambda(K_w - \bar{K}_w). \tag{31}$$

On the saddle path of the entire economy each relationship between capital in world economy and private consumption (or, aggregate consumption) is given by:

$$C_j - \bar{C}_j = -\frac{\bar{Z}_{hf}\bar{C}_w\bar{B}_{hf}\bar{\omega}_j}{\lambda(\bar{Z}_{hf}\bar{C}_w + \bar{\Omega} + \bar{\Omega}^*)}(K_w - \bar{K}_w), \tag{32a}$$

$$C - \bar{C} = -\frac{\bar{Z}_{hf}\bar{C}_w\bar{B}_{hf}\bar{\Omega}}{\lambda(\bar{Z}_{hf}\bar{C}_w + \bar{\Omega} + \bar{\Omega}^*)}(K_w - \bar{K}_w), \tag{32b}$$

$$C_w - \bar{C}_w = -\frac{\bar{Z}_{hf}\bar{C}_w\bar{B}_{hf}(\bar{\Omega} + \bar{\Omega}^*)}{\lambda(\bar{Z}_{hf}\bar{C}_w + \bar{\Omega} + \bar{\Omega}^*)}(K_w - \bar{K}_w). \tag{32c}$$

The $\bar{B}_{hf}$ is given by:

$$\bar{B}_{hf} = \frac{r'(\bar{p})\frac{\partial p}{\partial C_w}}{G_{C_w}}\left(\frac{f_i(k_i(\bar{p}))}{k_i(\bar{p}) - k_c(\bar{p})} - \delta - \lambda\right) + \frac{\lambda}{\bar{Z}_{hf}(2k_i(\bar{p}) - \bar{K}_w)}(>0), \tag{33}$$

where both countries have the same value of $\bar{B}_{hf}$.

3 Preferences and Wealth Distribution in the World Economy

We now examine the distribution of wealth in world economy. Then, our interests are to confirm the role of form of identical utility function for the dispersion of wealth. As in (23b), when the identical utility function is homothetic, the growth rate of private consumption among agents in both countries is the same. On the other hand, in the case of non-homothetic utility function, when the initially given dispersion of capital exists, the difference of initial capital held by each agent leads to the long-run level of private consumption, thereby being able to see that the growth rate of private consumption differs among agents, which may lead to a drastic change in determining the qualitative movement of wealth inequality over time. Hereafter, we simply express functions and variables unless they do not lead to the confusion. For instance, the production function $f_i(k_i(p(K_w, C_w)))$ simply is written as f_i.

Let us define the relative capital holding of agent j in home (or, foreign) country as follows: $\tilde{k}_{j,w} = 2k_j/K_w$ (or, $\tilde{k}^*_{j,w} = 2k^*_j/K_w$). Substituting the capital accumulation equations (12) and (14b) into $\dot{\tilde{k}}_{j,w} = 2\dot{k}_j/K_w - 2\tilde{k}_j\dot{K}_w/K_w^2$, we can show that

$$\dot{\tilde{k}}_{j,w} = \frac{2w(1-\tilde{k}_{j,w})}{K_w} + \frac{pC_w}{K_w}\left(\tilde{k}_{j,w} - \frac{2C_j}{C_w}\right), \tag{34a}$$

where the initial relative capital $\tilde{k}^0_{j,w}$ is given from the initial endowment. In addition, from (34a), the $\dot{\tilde{k}}_{j,w} = 0$ equation yields:

$$\bar{\tilde{k}}_{j,w} - \frac{2\bar{C}_j}{\bar{C}_w} = \frac{2\bar{w}(\bar{\tilde{k}}_{j,w} - 1)}{\bar{p}\bar{C}_w}. \tag{34b}$$

Next, substituting $w = p(f_c - k_c f'_c)$ into (34a), and furthermore, linearly approximating (34a) around the steady state, we can obtain:

$$\dot{\tilde{k}}_{j,w} = \rho(\tilde{k}_{j,w} - \bar{\tilde{k}}_{j,w}) + \frac{\bar{p}}{\bar{K}_w}\Bigg\{(\bar{\tilde{k}}_{j,w} - 1)\left(\frac{2\bar{w}}{\bar{p}\bar{C}_w}(C_w - \bar{C}_w) + 2\bar{k}_c f''_c \bar{k}'_c\left(\frac{\partial p}{\partial C_w}(C_w - \bar{C}_w) + \frac{\partial p}{\partial K_w}(K_w - \bar{K}_w)\right)\right)$$
$$- 2\bar{C}_j\left(\frac{C_j - \bar{C}_j}{\bar{C}_j} - \frac{C_w - \bar{C}_w}{\bar{C}_w}\right)\Bigg\}, \tag{34c}$$

where we use (34b) and $\bar{C}_w - \frac{2\bar{w}}{\bar{p}} = \frac{\rho\bar{K}_w}{\bar{p}}$ from $\dot{K}_w = 0$ equation (14b).

Substituting (30) into (34c), we can show that

$$\dot{\tilde{k}}_{j,w} = \rho(\tilde{k}_{j,w} - \bar{\tilde{k}}_{j,w}) + (\bar{\tilde{k}}_{j,w} - 1)\frac{\bar{D}_{hf}(K_w - \bar{K}_w)}{\bar{K}_w} - \frac{2\bar{C}_j\bar{p}}{\bar{K}_w}\left(\frac{C_j - \bar{C}_j}{\bar{C}_j} - \frac{C_w - \bar{C}_w}{\bar{C}_w}\right), \tag{35}$$

where the variable $\bar{D}_{hf}$ is given by:

$$\bar{D}_{hf} = 2\bar{p}\left\{\frac{\bar{w}}{\bar{p}\bar{C}_w}\left(-\frac{G_{K_w}-\delta-\lambda}{G_{C_w}}\right)+\bar{k}_c f_c''\bar{k}_c'\left(\frac{\partial p}{\partial C_w}\left(-\frac{G_{K_w}-\delta-\lambda}{G_{C_w}}+\frac{\partial p}{\partial K_w}\right)\right)\right\}. \quad (36a)$$

Furthermore, assuming that the production structure in consumption and investment sectors are respectively given by the Cobb-Douglas type $f_c(k_c) = k_c^{\alpha_c}$ and $f_i(k_i) = k_i^{\alpha_i}$, $(36a)$ can be rewritten as:[4]

$$\bar{D}_{hf} = \frac{2\bar{p}\bar{w}\bar{k}_i'}{G_{C_w}\bar{Z}_{hf}\bar{C}_w(\bar{k}_i-\bar{k}_c)} \times \frac{\alpha_c(1-\alpha_i)(\delta+\rho)}{\alpha_i(1-\alpha_c)(\delta+\rho-\alpha_i\delta)}\left[\lambda+\alpha_i(1-\alpha_i)\delta\right]. \quad (36b)$$

Then, the sign of $\bar{D}_{hf}$ is determined by $[\lambda+\alpha_i(1-\alpha_i)\delta]$. In detail, if the speed of convergence is fast so that $\alpha_i(1-\alpha_i)\delta < -\lambda$, then $\bar{D}_{hf}$ has a positive sign; and alternatively, if its speed is slow such that $\alpha_i(1-\alpha_i)\delta > -\lambda$, then $\bar{D}_{hf}$ has a negative sign.

Therefore, as for the sign of $\bar{D}_{hf}$, we can summarize the results.

Result 1. *Suppose that the production function is specified by Cobb-Douglas type, $f_c = k_c^{\alpha_c}$ and $f_i = k_i^{\alpha_i}$. In addition, assume a growing economy $\bar{K}_w > K_w^0$ under Assumption 1. Then, if $\alpha_i(1-\alpha_i)\delta < (>) - \lambda$, the sign of $\bar{D}_{hf}$ is positive (negative).*

3.1 Homothetic Utility Function

Whether the identical utility function is homothetic or non-homothetic is important to characterize the wealth distribution. This is because under the identical and homothetic utility function, the (pure) elasticities of intertemporal substitution, defined by ω_j/C_j, Ω/C and $(\Omega+\Omega^*)/C_w$, are the same, thereby being able to see that from $(32a)$, $(32b)$ and $(32c)$, the equations $\frac{C_j-\bar{C}_j}{\bar{C}_j} = \frac{C-\bar{C}}{\bar{C}} = \frac{C_w-\bar{C}_w}{\bar{C}_w}$ are held. Under the assumption that the utility function is homothetic, the equation (35) is simplified as follows:

$$\dot{\tilde{k}}_{j,w} = \rho(\tilde{k}_{j,w}-\bar{\tilde{k}}_{j,w}) + (\bar{\tilde{k}}_{j,w}-1)\frac{\bar{D}_{hf}(K_w-\bar{K}_w)}{\bar{K}_w}, \quad (37a)$$

which leads to the following:

$$\tilde{k}_{j,w} = \bar{\tilde{k}}_{j,w} + (\bar{\tilde{k}}_{j,w}-1)\frac{\bar{D}_{hf}}{\bar{K}_w}\frac{\bar{K}_w-K_w^0}{\rho-\lambda}e^{\lambda t}. \quad (37b)$$

Now, when we define the dispersion of individual wealth as $\Sigma_j \equiv (\tilde{k}_{j,w}-1)$, the dynamic equation of relative wealth, $(37b)$, is given by:

$$\Sigma_j = \bar{\Sigma}_j\left(1+\frac{\bar{D}_{hf}}{\bar{K}_w}\frac{\bar{K}_w-K_w^0}{\rho-\lambda}e^{\lambda t}\right), \quad (38a)$$

[4] See Appendix B.

where at the initial period ($t = 0$), the steady-state dispersion of individual wealth is related as the initially given dispersion of individual wealth:

$$\bar{\Sigma}_j = \frac{\Sigma_j^0}{1 + \frac{\bar{D}_{hf}}{\bar{K_w}} \frac{\bar{K}_w - K_w^0}{\rho - \lambda}}. \tag{38b}$$

In addition, the dynamic motion of dispersion of individual wealth is characterized by

$$\dot{\Sigma}_j = \lambda \bar{\Sigma}_j \frac{\bar{D}_{hf}}{\bar{K_w}} \frac{\bar{K}_w - K_w^0}{\rho - \lambda} e^{\lambda t} \tag{38c}$$

In the case of homothetic utility function, the growth rate of consumption, $\dot{C}_j/C_j$ is the same among agents as confirmed in ($23b$), which means that both the level of capital held by an agent j and the average level of capital in each country similarly grows. That is, since the catching-up of capital does not arise, it holds that $\bar{\Sigma}_j$ and Σ_j^0 have an identical sign. Therefore, from ($38b$) we can obtain:

$$1 + \frac{\bar{D}_{hf}}{\bar{K_w}} \frac{\bar{K}_w - K_w^0}{\rho - \lambda} > 0. \tag{39}$$

Noting that the stable root λ has a negative sign, it holds that $\text{sign}\left(\dot{\Sigma}_j\right) = \text{sign}\left(\bar{D}_{hf}\bar{\Sigma}_j\right)$. To give the intuitive explanation, for instance, let us suppose that $\bar{D}_{hf}$ has a positive sign. Then, if the sign of $\bar{\Sigma}_j$ is positive, it means that $\dot{\Sigma}_j < 0$, thereby being able to estimate that $\Sigma_j^0 > \bar{\Sigma}_j(> 0)$. That is, the difference between the agent j' capital k_j and the average level of aggregate capital in a country $K_w/2$ shrinks over time in that the level of individual capital k_j approaches to the average level $K_w/2$ along time. On the other hand, when $\bar{\Sigma}_j$ has a negative sign, it holds that $\Sigma_j^0 < \bar{\Sigma}_j(< 0)$, which can be concluded that the difference between the agent j' capital and the average level of aggregate capital in a country becomes larger. When $\bar{D}_{hf}$ has a negative sign, the opposite relationship can be applicable.

We now define the distribution of wealth as follows:

$$S_w = s_w + s_w^*, \quad \text{where } s_w = \int_0^1 (\Sigma_j)^2 dj, \quad s_w^* = \int_0^1 (\Sigma_j^*)^2 dj, \tag{40}$$

and then, from (38a) and (38b), we can show that[5]

$$s_w = \left(\frac{1 + \frac{\bar{D}_{hf}}{\bar{K}_w} \frac{\bar{K}_w - K_w^0}{\rho - \lambda} e^{\lambda t}}{1 + \frac{\bar{D}_{hf}}{\bar{K}_w} \frac{\bar{K}_w - K_w^0}{\rho - \lambda}} \right)^2 s_w^0. \tag{41}$$

Noting that s_w in home country and s_w^* in foreign country are the same except for the difference of initial distribution of wealth, from (40) we can write the wealth inequality in world economy as:

$$S_w = \left(\frac{1 + \frac{\bar{D}_{hf}}{\bar{K}_w} \frac{\bar{K}_w - K_w^0}{\rho - \lambda} e^{\lambda t}}{1 + \frac{\bar{D}_{hf}}{\bar{K}_w} \frac{\bar{K}_w - K_w^0}{\rho - \lambda}} \right)^2 S_w^0, \tag{42a}$$

and furthermore, the long-run level of wealth inequality is given by:

$$\bar{S}_w = \frac{S_w^0}{\left(1 + \frac{\bar{D}_{hf}}{\bar{K}_w} \frac{\bar{K}_w - K_w^0}{\rho - \lambda}\right)^2}. \tag{42b}$$

Then, the results given in (42b) can be summarized as follows.

Proposition 2 *Suppose that the identical utility function is homothetic. Moreover, assume that the economy is growing $\bar{K}_w > K_w^0$. The positive (negative) sign of $\bar{D}_{hf}$ leads to $\bar{S}_w < (>) S_w^0$.*

Turning back to Result 1, we can argue that when the speed of convergence is fast enough so that $\bar{D}_{hf}$ has a positive sign, the initial level of wealth inequality is larger than the steady-state level. Following García-Peñalosa and Turnovsky (2008) and Mino and Nakamoto (2015), the result is reasonable. Suppose that the speed of convergence is fast enough in a growing economy $\bar{K}_w > K_w^0$. Then, a fast increase in the level of aggregate capital leads to a fast decrease in the return to capital. Because such a negative impact largely affects the consumption-saving decision of agents who have more wealth, such a fast speed of convergence makes the long-run wealth inequality smaller. On the other hand, if the speed of convergence is small such that $\bar{D}_{hf}$ has a negative sign, the opposite relationship can be seen.

[5] Substituting (38b) into (38a), we make use of the following.

$$\Sigma_j = \frac{1 + \frac{\bar{D}_{hf}}{\bar{K}_w} \frac{\bar{K}_w - K_w^0}{\rho - \lambda} e^{\lambda t}}{1 + \frac{\bar{D}_{hf}}{\bar{K}_w} \frac{\bar{K}_w - K_w^0}{\rho - \lambda}} \Sigma_j^0.$$

3.2 Non-homothetic Utility Function

We turn to the case in which the utility function is identical among agents but non-homothetic. In this case, because of the difference of initially given capital, the (pure) elasticities of intertemporal substitution, ω_j/C_j, Ω/C and $(\Omega+\Omega^*)/C_w$, are not the same, which estimates that the dynamic behavior of relative wealth as well as the dispersion of wealth is more complicated.

From (30) and ($32a$) we firstly consider the difference of linear approximation between private consumption and aggregate consumption:

$$\frac{C_j-\bar{C}_j}{\bar{C}_j}-\frac{C_w-\bar{C}_w}{\bar{C}_w}=-\frac{G_{K_w}-\delta-\lambda}{G_{C_w}(\bar{\Omega}+\bar{\Omega}^*)}\left(\frac{\omega_j}{\bar{C}_j}-\frac{\bar{\Omega}+\bar{\Omega}^*}{\bar{C}_w}\right)(K_w-\bar{K}_w). \tag{43}$$

Since the difference of elasticities of intertemporal substitution, $\left(\frac{\omega_j}{\bar{C}_j}-\frac{\bar{\Omega}+\bar{\Omega}^*}{\bar{C}_w}\right)$, exists, we can see that $\frac{C_j-\bar{C}_j}{\bar{C}_j}\neq\frac{C_w-\bar{C}_w}{\bar{C}_w}$ unlike the homothetic utility function.

Substituting (43) into (35), we can confirm the dynamics of relative wealth as follows:

$$\dot{\tilde{k}}_{j,w}=\rho(\tilde{k}_{j,w}-\bar{\tilde{k}}_{j,w})+\left((\bar{\tilde{k}}_{j,w}-1)\frac{\bar{D}_{hf}}{\bar{K}_w}+\frac{2\bar{C}_j\bar{p}(G_{K_w}-\delta-\lambda)}{\bar{K}_wG_{C_w}(\bar{\Omega}+\bar{\Omega}^*)}\left(\frac{\omega_j}{\bar{C}_j}-\frac{\bar{\Omega}+\bar{\Omega}^*}{\bar{C}_w}\right)\right)(K_w-\bar{K}_w), \tag{44a}$$

and then, as in the last subsection, using $\Sigma_j=\tilde{k}_{j,w}-1$ and arranging for the differential equation, we obtain the stable path of relative wealth:

$$\Sigma_j=\left(1+\frac{\bar{D}_{hf}(\bar{K}_w-K_w^0)}{\bar{K}_w(\rho-\lambda)}e^{\lambda t}\right)\bar{\Sigma}_j+\frac{2\bar{C}_j\bar{p}(G_{K_w}-\delta-\lambda)}{\bar{K}_wG_{C_w}(\bar{\Omega}+\bar{\Omega}^*)}\left(\frac{\bar{\omega}_j}{\bar{C}_j}-\frac{\bar{\Omega}+\bar{\Omega}^*}{\bar{C}_w}\right)\frac{\bar{K}_w-K_w^0}{\rho-\lambda}e^{\lambda t}, \tag{44b}$$

where substituting $t=0$ into ($44b$) yields:

$$\bar{\Sigma}_j=\frac{\Sigma_j^0-\frac{2\bar{C}_j\bar{p}(G_{K_w}-\delta-\lambda)}{\bar{K}_wG_{C_w}(\bar{\Omega}+\bar{\Omega}^*)}\left(\frac{\bar{\omega}_j}{\bar{C}_j}-\frac{\bar{\Omega}+\bar{\Omega}^*}{\bar{C}_w}\right)\frac{\bar{K}_w-K_w^0}{\rho-\lambda}e^{\lambda t}}{1+\frac{\bar{D}_{hf}(\bar{K}_w-K_w^0)}{\bar{K}_w(\rho-\lambda)}}. \tag{44c}$$

Now, let us differentiate ($44b$) with respect to time:

$$\dot{\Sigma}_j=\lambda\bar{\Sigma}_j\frac{\bar{D}_{hf}(\bar{K}_w-K_w^0)}{\bar{K}_w(\rho-\lambda)}e^{\lambda t}+\lambda\frac{2\bar{C}_j\bar{p}(G_{K_w}-\delta-\lambda)}{\bar{K}_wG_{C_w}(\bar{\Omega}+\bar{\Omega}^*)}\left(\frac{\bar{\omega}_j}{\bar{C}_j}-\frac{\bar{\Omega}+\bar{\Omega}^*}{\bar{C}_w}\right)\frac{\bar{K}_w-K_w^0}{\rho-\lambda}e^{\lambda t}. \tag{45}$$

where note that $(G_{K_w}-\delta-\lambda)>0$ and $G_{C_w}<0$.

From the comparison between homotheticity and non-homotheticity of utility functions, we can see the following two points in the case of non-homothetic utility function. First, from (45) it is evident that not only the sign of $\bar{D}_{hf}$ but also that of $(\frac{\bar{\omega}_j}{\bar{C}_j}-\frac{\bar{\Omega}+\bar{\Omega}^*}{\bar{C}_w})$ have qualitative impacts on the dynamic motion of relative wealth. For instance, suppose that $\bar{\Sigma}_j$ has a positive sign. Then, if $\bar{D}_{hf} > 0$, from the last subsection the relative wealth decreases in the case of homothetic utility function. Alternatively, in the case of non-homothetic utility function, the positive sign of $\bar{D}_{hf}$ does not necessarily lead to the same conclusion. In detail, when the condition $\frac{\bar{\omega}_j}{\bar{C}_j} < \frac{\bar{\Omega}+\bar{\Omega}^*}{\bar{C}_w}$ is additionally provided, it is sufficient to hold that $\dot{\Sigma}_j < 0$, which leads to $\Sigma_j^0 > \bar{\Sigma}_j > 0$. Intuitively, since the smaller elasticity of intertemporal substitution of agent j means that he does not save with enthusiasm, in a growing economy his capital does not grow faster than the average capital. As a result, when the speed of convergence is fast such that $\bar{D}_{hf} > 0$, the difference of capital between agent j and average agent in world economy shrinks.

Next, even if the inequality (39) is satisfied, it cannot be necessarily held that $\bar{\Sigma}_j$ and Σ_j^0 have the same sign, meaning that the possibility of catching-up cannot be excluded unlike the homothetic utility function. Concretely, assume that $\Sigma_j^0 > 0$ under (39). Then, from (44c) we can confirm that if $(\bar{\Omega}+\bar{\Omega}^*)/\bar{C}_w > \bar{\omega}_j/\bar{C}_j$, it may hold that $\bar{\Sigma}_j < 0$. In other words, even if an agent j is relatively wealth-rich at the initial economy, the smaller elasticity of intertemporal substitution may lend to the relatively inferior position of wealth in the long run in the sense that $\bar{\Sigma}_j < 0$. Since the capital monotonically grows in a growing economy, it means that the level of capital held by the agent j is caught by the average level of individual capital during the transition, $\Sigma_j = 0$.

Making use of the definition of wealth inequality in (40), we can show that[6]

$$s_w = \left(\frac{1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho-\lambda)} e^{\lambda t}}{1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho-\lambda)}} \right)^2 s_w^0 - \frac{2\left(1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho-\lambda)} e^{\lambda t}\right)(1 - e^{\lambda t})}{\left(1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho-\lambda)}\right)^2} \int_0^1 A_j \Sigma_j^0 dj$$
$$+ \left(\frac{1 - e^{\lambda t}}{1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho-\lambda)}} \right)^2 \int_0^1 A_j^2 dj, \tag{46a}$$

where

$$A_j = \frac{2\bar{C}_j \bar{p}(G_{K_w} - \delta - \lambda)}{\bar{K}_w G_{C_w}(\bar{\Omega}+\bar{\Omega}^*)} \left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega}+\bar{\Omega}^*}{\bar{C}_w} \right) \frac{\bar{K}_w - K_w^0}{\rho - \lambda},$$

[6] To derive (46a), from (44b) and (44c) we make use of the following:

$$\Sigma_j = \frac{\left(1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho-\lambda)} e^{\lambda t}\right)(\Sigma_j^0 - A_j)}{1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho-\lambda)}} + A_j e^{\lambda t}.$$

$$\int_0^1 A_j \Sigma_j^0 dj = \frac{2\bar{p}(G_{K_w} - \delta - \rho)(\bar{K}_w - K_w^0)}{\bar{K}_w G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)(\rho - \lambda)} \int_0^1 \bar{C}_j \left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega} + \bar{\Omega}^*}{\bar{C}_w} \right) \Sigma_j^0 dj,$$

$$\int_0^1 A_j^2 dj = \left(\frac{2\bar{p}(G_{K_w} - \delta - \rho)(\bar{K}_w - K_w^0)}{\bar{K}_w G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)(\rho - \lambda)} \right)^2 \int_0^1 \bar{C}_j^2 \left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega} + \bar{\Omega}^*}{\bar{C}_w} \right)^2 dj (> 0).$$

Consequently, under the non-homothetic utility function, the wealth inequality in world economy is given by:

$$S_w = \left(\frac{1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho - \lambda)} e^{\lambda t}}{1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho - \lambda)}} \right)^2 S_w^0 - \frac{2\left(1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho - \lambda)} e^{\lambda t}\right)(1 - e^{\lambda t})}{\left(1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho - \lambda)}\right)^2}$$
$$\times \left(\int_0^1 (A_j \Sigma_j^0 + A_j^* \Sigma_j^{*,0}) dj \right) + \left(\frac{1 - e^{\lambda t}}{1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho - \lambda)}} \right)^2 \left(\int_0^1 (A_j^2 + (A_j^*)^2) dj \right). \tag{46b}$$

Substituting $t = \infty$ into $(46b)$, we can obtain the following:

$$\bar{S}_w = \frac{1}{\left(1 + \frac{\bar{D}_{hf}(\bar{K}_w - K_w^0)}{\bar{K}_w(\rho - \lambda)}\right)^2} \left(S_w^0 + \underbrace{\left(\int_0^1 (A_j^2 + (A_j^*)^2) dj \right)}_{(\#1)} - 2 \underbrace{\left(\int_0^1 (A_j \Sigma_j^0 + A_j^* \Sigma_j^{*,0}) dj \right)}_{(\#2)} \right). \tag{46c}$$

Then, from $(46c)$ we can summarize the sufficient conditions with respect to the relationship of wealth inequality between the long run economy and the initial one.

Proposition 3 *Suppose that the identical utility function is non-homothetic. Moreover, assume that the inequality* (39) *is satisfied in a growing economy $\bar{K}_w > K_w^0$. (i) Assume that $\bar{D}_{hf} > 0$. It holds that $\bar{S}_w < S_w^0$ if the following inequality is satisfied:*

$$2 \left(\int_0^1 (A_j \Sigma_j^0 + A_j^* \Sigma_j^{*,0}) dj \right) > \int_0^1 (A_j^2 + (A_j^*)^2) dj. \tag{47a}$$

(ii) Assume that $\bar{D}_{hf} < 0$. It holds that $\bar{S}_w > S_w^0$ if either inequality is satisfied:

$$\int_0^1 (A_j^2 + (A_j^*)^2) dj > 2 \left(\int_0^1 (A_j \Sigma_j^0 + A_j^* \Sigma_j^{*,0}) dj \right) > 0, \tag{47b}$$

$$\int_0^1 (A_j \Sigma_j^0 + A_j^* \Sigma_j^{*,0}) dj < 0. \tag{47c}$$

Proof. It can be easily seen from $(46c)$. ■

Comparing $(42b)$ with $(46c)$, we can easily see that the non-homotheticity of utility function is newly added two impacts on the wealth inequality. This is because the long-run elasticities of intertemporal substitution, $\bar{\omega}_j/\bar{C}_j$, $\bar{\Omega}/\bar{C}$ and $(\bar{\Omega}+\bar{\Omega}^*)/\bar{C}_w$ differ each other. As a result, the conditions in Proposition 4 are rather complex, but we may give an intuitive implication.

Then, we are interested in the newly added terms (#1) and (#2). The (#1) in (46c) indicates the dispersion of elasticities of intertemporal substitution in the long run. For instance, if the dispersion of elasticities of intertemporal substitution is large in home or foreign country, then the value of (#1) is large, which means that more dispersion of elasticities of intertemporal substitution makes the long-run wealth inequality larger. Next, we turn to the term (#2) in (46c), which shows a correlation between the initial holdings of capital stocks and the difference of elasticities of intertemporal substitution between an agent j and the average agent in world economy. If the initially relative-wealth rich has the greater elasticity of intertemporal substitution in each country on average, it holds that $\int_0^1 \bar{C}_j \left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega}+\bar{\Omega}_j}{\bar{C}_w}\right) \Sigma_j^0 dj > 0$ and $\int_0^1 \bar{C}_j \left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega}+\bar{\Omega}_j}{\bar{C}_w}\right) \Sigma_j^0 dj > 0$. As a result, (#2) has a negative sign, so that it may be seen that $\bar{S}_w > S_w^0$. This is because the initially relative-wealth riches make more saving during the transition, so that they have more wealth in the long run.

We now confirm the sufficient conditions in $(47a)-(47c)$. First, the conditions (47a) and (47b) show whether the impact in (#1) is larger than that in (#2) or not. If the positive impact given by (#1) is smaller than that by (#2) under the negative sign of $\bar{D}_{hf}$, then the initially given dispersion of wealth is larger than the long-run level. When we focus on the positive sign of (#2), this case means that the initially relative-wealth riches have smaller elasticities of intertemporal substitution in private consumption on average, implying that since they dislike saving, the levels of their capital approach to the average level of capital in world economy. On the other hand, when the impact given by (#1) is larger than that by (#2) and $\bar{D}_{hf} > 0$, the initial distribution of wealth is smaller than the long-run level. In particular, as for the case (ii), if the condition (47c) is satisfied under the assumption that $\bar{D}_{hf} < 0$, we can see that $\bar{S}_w > S_w^0$.

As for these cases (i) and (ii) in this proposition, it should be noted that the relationship between $\bar{D}_{hf} >$ (or < 0) and the inequality $\bar{S}_w <$ (or $>$)S_w^0 under the non-homothetic utility function does not deviate from that under the homothetic utility function. Therefore, the qualitatively seen results do not change. Alternatively, considering that these conditions in cases (i) and (ii) are sufficient but not necessary, this proposition says that when the sufficient conditions are not satisfied, the non-homotheticity of utility function may deviate from the existing finding that the fast (slow) speed of convergence leads to an expansion (a reduction) in wealth inequality.

4 Wealth Distribution in a Country

We now turn to the domestic distribution of wealth. Our interests are analytically to see whether the dynamic behavior and the steady-state characterization of domestic distribution of wealth qualitatively deviates from those observed in world economy.

4.1 Homothetic Utility Function

Defining the relative capital between an agent j and the average agent in domestic economy by $\tilde{k}_j = k_j/K$, we can show that

$$\dot{\tilde{k}}_j = \rho(\tilde{k}_j - \bar{\tilde{k}}_j) + (\bar{\tilde{k}}_j - 1)\frac{\bar{D}_{hf}}{2\bar{K}}(K_w - \bar{K}_w), \tag{48}$$

and, as in the last section, the stable solution for the time path of the relative capital is given by:

$$\sigma_j = \bar{\sigma}_j\left(1 + \frac{\bar{D}_{hf}}{2\bar{K}}\frac{\bar{K}_w - K_w^0}{\rho - \lambda}e^{\lambda t}\right), \tag{49}$$

which leads to the dynamic motion of relative wealth in domestic economy as follows:

$$\dot{\sigma}_j = \lambda\bar{\sigma}_j\frac{\bar{D}_{hf}}{2\bar{K}}\frac{\bar{K}_w - K_w^0}{\rho - \lambda}e^{\lambda t}. \tag{50}$$

From (50), we can confirm that the dynamic motion of relative wealth in domestic economy depends on the sign of $\bar{\sigma}_j\bar{D}_{hf}$. Supposing that an agent j is relatively wealth-rich in the long run $\bar{\sigma}_j > 0$, the positive sign of $\bar{D}_{hf}$ provides $\dot{\sigma}_j < 0$. In sum, in the case of homothetic utility function, the dynamic characterization of relative wealth in domestic economy is the same with that in world economy in the sense that when the speed of convergence is fast such that $\bar{D}_{hf} > 0$, the level of capital held by the agent j approaches to the average level per person in domestic economy during the transition. Alternatively, if $\bar{D}_{hf}$ has a negative sign, it holds that $\dot{\sigma}_j > 0$.

The above results mean that when we define the index of wealth inequality in domestic economy by

$$S = \int_0^1 \sigma_j^2 dj, \tag{51}$$

then in the case of homothetic utility function, the dynamic motion of wealth inequality is given by:

$$S = \left(\frac{1 + \frac{\bar{D}_{hf}}{2\bar{K}} \frac{\bar{K}_w - K_w^0}{\rho - \lambda} e^{\lambda t}}{1 + \frac{\bar{D}_{hf}}{2\bar{K}} \frac{\bar{K}_w - K_w^0}{\rho - \lambda}} \right)^2 S^0, \tag{52a}$$

and then, the steady-state level of wealth inequality is:

$$\bar{S} = \frac{S^0}{\left(1 + \frac{\bar{D}_{hf}}{2\bar{K}} \frac{\bar{K}_w - K_w^0}{\rho - \lambda}\right)^2}. \tag{52b}$$

As a result, we can argue the following.

Proposition 4 *Suppose that the identical utility function is homothetic. Moreover, assume that the economy is growing $\bar{K}_w > K_w^0$. The positive (negative) sign of $\bar{D}_{hf}$ leads to $\bar{S} < (>)S^0$.*

Proposition 4 says that under the homothetic utility function, whether the speed of convergence is fast or slow determines the qualitatively observed characterization of wealth inequality in domestic economy as in Proposition 2, meaning that when the identical utility function is assumed to be homothetic, the steady-state characterization of wealth inequality in domestic economy is qualitatively the same with that in world economy.

4.2 Non-homothetic Utility Function

Now, let us suppose that the identical utility function is non-homothetic. We derive the dynamic equation of relative wealth in domestic economy:

$$\begin{aligned}\dot{\tilde{k}}_j = \rho(\tilde{k}_j - \bar{\tilde{k}}_j) + \Bigg\{ \frac{\bar{\tilde{k}}_j - 1}{\bar{K}} &\left(\frac{\bar{D}_{hf}}{2} - \frac{\bar{w}}{\bar{p}} \left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \underbrace{\left(\frac{\bar{\Omega}}{\bar{C}} - \frac{\bar{\Omega} + \bar{\Omega}^*}{\bar{C}_w} \right)}_{(\#3)} \right) \\ + \frac{\bar{C}_j \bar{p}}{\bar{K}} &\left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \underbrace{\left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega}}{\bar{C}} \right)}_{(\#4)} \Bigg\}.\end{aligned} \tag{53}$$

Unlike the case of homothetic utility function, the non-homotheticity leads to the newly two terms (#3) and (#4). First, the part (#3) expresses the difference of elasticities of intertemporal substitution between the average agent in home country and the average agent in world economy. Next, (#4) shows the difference of elasticities of intertemporal substitution between an agent j and the average agent in home country.

Solving the dynamic equation, we can see the following:

$$\begin{aligned}\sigma_{j,k} = \bar{\sigma}_{j,k} + \Bigg\{ & \frac{\bar{\sigma}_{j,k}}{\bar{K}} \left(\frac{\bar{D}_{hf}}{2} - \frac{\bar{w}}{\bar{p}} \left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \left(\frac{\bar{\Omega}}{\bar{C}} - \frac{\bar{\Omega} + \bar{\Omega}^*}{\bar{C}_w} \right) \right) \\ & + \frac{\bar{C}_j \bar{p}}{\bar{K}} \left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega}}{\bar{C}} \right) \Bigg\} \frac{\bar{K}_w - K_w^0}{\rho - \lambda} e^{\lambda t}, \end{aligned} \tag{54a}$$

which gives the following:

$$\begin{aligned}\dot{\sigma}_{j,k} = \lambda \Bigg\{ & \frac{\bar{\sigma}_{j,k}}{\bar{K}} \left(\frac{\bar{D}_{hf}}{2} - \frac{\bar{w}}{\bar{p}} \left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \left(\frac{\bar{\Omega}}{\bar{C}} - \frac{\bar{\Omega} + \bar{\Omega}^*}{\bar{C}_w} \right) \right) \\ & + \frac{\bar{C}_j \bar{p}}{\bar{K}} \left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega}}{\bar{C}} \right) \Bigg\} \frac{\bar{K}_w - K_w^0}{\rho - \lambda} e^{\lambda t}. \end{aligned} \tag{54b}$$

Then, the dynamic behavior of relative wealth in domestic economy is more complicated in the case of non-homothetic utility function. For instance, suppose that an agent j in home country is relatively wealth-rich in the steady state ($\bar{\sigma}_{j,k} > 0$), and furthermore that $\bar{D}_{hf}$ has a positive sign. If the utility function is homothetic, then it holds that $\dot{\sigma}_{j,k} < 0$. However, in the case of non-homothetic utility function, even if $\bar{D}_{hf} > 0$, it does not necessarily seen that $\dot{\sigma}_{j,k} < 0$ because of the difference of long-run elasticities of intertemporal substitution. For example, when either $\bar{\Omega}/\bar{C} < (\bar{\Omega} + \bar{\Omega}^*)/\bar{C}_w$ or $\bar{\Omega}/\bar{C} < \bar{\omega}_j/\bar{C}_j$ is satisfied, we may confirm that $\dot{\sigma}_{j,k} > 0$. In other words, if the average agent in home country does not like to save compared with the average agent in world economy (i.e., the average agent in foreign country) or an agent j, the level of capital held by an agent j rapidly grows relative to the average agent in home country. In particular, as for the inequality $\bar{\Omega}/\bar{C} < (\bar{\Omega} + \bar{\Omega}^*)/\bar{C}_w$, it can be said that the existence of foreign country may have a qualitatively pivotal role in characterizing the relative wealth in home country.

Following the same procedure so far, we characterize the relative wealth at time t:[7]

$$\sigma_{j,k} = \frac{(1 + F e^{\lambda t})(\sigma_{j,k}^0 - E_j)}{1 + F} + E_j e^{\lambda t} \tag{55}$$

[7]We make use of the following:

$$\bar{\sigma}_{j,k} = \frac{\sigma_{j,k}^0 - \frac{\bar{C}_j \bar{p}}{\bar{K}} \left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega}}{\bar{C}} \right) \frac{\bar{K}_w - K_w^0}{\rho - \lambda}}{1 + \frac{1}{\bar{K}} \left(\frac{\bar{D}_{hf}}{2} - \frac{\bar{w}}{\bar{p}} \left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \left(\frac{\bar{\Omega}}{\bar{C}} - \frac{\bar{\Omega} + \bar{\Omega}^*}{\bar{C}_w} \right) \right) \frac{\bar{K}_w - K_w^0}{\rho - \lambda}}$$

where note that E_j is different among agents but F is common as follows:

$$E_j \equiv \frac{\bar{C}_j \bar{p}}{\bar{K}} \left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega}}{\bar{C}} \right) \frac{\bar{K}_w - K_w^0}{\rho - \lambda}. \tag{56a}$$

$$F \equiv \frac{1}{\bar{K}} \left(\frac{\bar{D}_{hf}}{2} - \frac{\bar{w}}{\bar{p}} \left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \left(\frac{\bar{\Omega}}{\bar{C}} - \frac{\bar{\Omega} + \bar{\Omega}^*}{\bar{C}_w} \right) \right) \frac{\bar{K}_w - K_w^0}{\rho - \lambda}. \tag{56b}$$

Therefore, we lead to the following:

$$S = \left(\frac{1 + Fe^{\lambda t}}{1 + F} \right)^2 S^0 - 2 \left(\frac{(1 + Fe^{\lambda t})(1 - e^{\lambda t})}{(1 + F)^2} \right) \int_0^1 \sigma_{j,k}^0 E_j dj + \left(\frac{1 - e^{\lambda t}}{1 + F} \right)^2 \int_0^1 E_j^2 dj, \tag{57a}$$

and furthermore, substituting $t = \infty$ into $(57a)$, we can show the steady-state level of domestic distribution of wealth:

$$\bar{S} = \frac{1}{(1 + F)^2} \left\{ S^0 + \underbrace{\int_0^1 E_j^2 dj}_{(\#5)} - 2 \underbrace{\int_0^1 \sigma_{j,k}^0 E_j dj}_{(\#6)} \right\}, \tag{57b}$$

where

$$\int_0^1 E_j^2 dj = \left(\frac{\bar{p}}{\bar{K}} \left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \frac{\bar{K}_w - K_w^0}{\rho - \lambda} \right)^2 \int_0^1 \bar{C}_j^2 \left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega}}{\bar{C}} \right)^2 dj (> 0),$$

$$\int_0^1 \sigma_{j,k}^0 E_j dj = \frac{\bar{p}}{\bar{K}} \left(\frac{G_{K_w} - \delta - \lambda}{G_{C_w}(\bar{\Omega} + \bar{\Omega}^*)} \right) \frac{\bar{K}_w - K_w^0}{\rho - \lambda} \int_0^1 \bar{C}_j \left(\frac{\bar{\omega}_j}{\bar{C}_j} - \frac{\bar{\Omega}}{\bar{C}} \right) \sigma_{j,k}^0 dj.$$

As in world economy, the introduction of non-homotheticity of utility function includes new impacts caused by the difference of long-run elasticities of intertemporal substitution. Taking account of (#1), (#2), (#5) and (#6), we can see that the newly given impacts (#1) and (#5) show the dispersion of elasticities of intertemporal substitution, and that (#2) and (#6) indicates to a correlation between the initial holdings of capital stocks and the difference of elasticities of intertemporal substitution. In that regard, these impacts have the similar implication in domestic and world economies.

However, we must notice that the dispersion of elasticities of intertemporal substitution in (#1) and (#2) is based on the average elasticity of intertemporal substitution in world economy; and alternatively, the dispersion in (#5) and (#6) is based on the average elasticity in home country, which expects the following case. To understand the intuition simply, let us consider an extreme case. Now, we suppose that the long-run dispersion of elasticities of intertemporal substitution in home country does not exist in the sense that $\bar{\omega}_j/\bar{C}_j = \bar{\Omega}/\bar{C}$

for all agents j in home country, which means that (#5) and (#6) are zero. As a result, the long-run level of wealth inequality is:

$$\bar{S} = \frac{S^0}{(1+F)^2}, \tag{58}$$

which implies that if $F > (<)0$, then $\bar{S} < (>)S^0$. However, the existence of foreign country allows us to see that (#1) and (#2) are not zero. Then, we expect that the dynamic motion and the steady-state characterization of wealth inequality in home country may not be faithfully the same with those in world economy.

Finally, we must notice that the part F in the denominator of ($57b$) includes the difference of elasticities of interetemporal substitution between home country and world. Therefore, even if $\bar{D}_{hf}$ has a positive sign, the sign of F may be negative. Turning back to (58), we understand this case. Suppose that the speed of convergence is fast in the sense that $\bar{D}_{hf}$ has a positive sign. But, if the average elasticity of intertemporal substitution in home country, $\bar{\Omega}/\bar{C}$, is quite small relative to $(\bar{\Omega}+\bar{\Omega}^*)/\bar{C}_w$, then F may have a negative sign, and hence, it holds that $\bar{S}_w > S_w^0$.

Appendices

Appendix A

Notice that the steady-state value of relative price is uniquely determined in $(28b)$. Therefore, totally differentiating (16) given price, we can show that

$$\bar{C}_w = C_w(\bar{K}_w), \tag{A.1}$$

where

$$\frac{\partial \bar{C}_w}{\partial \bar{K}_w} = -\frac{f_c}{\bar{k}_i - \bar{k}_c}(< 0), \ \lim_{\bar{K}_w \to 0} \bar{C}_w = \frac{2\bar{k}_i f_c}{\bar{k}_i - \bar{k}_c}(> 0), \ \lim_{\bar{K}_w \to \infty} \bar{C}_w = -\infty(< 0).$$

Next, substituting $(A.1)$ into $\dot{K}_w = 0$ yields:

$$\Psi(\bar{K}_w) \equiv G(\bar{K}_w, \bar{C}_w(\bar{K}_w)) - \delta \bar{K}_w = 0. \tag{A.2}$$

where[8]

$$\Psi'(\bar{K}_w) = \frac{f_i}{\bar{k}_i - \bar{k}_c} - \delta(> 0),$$
$$\lim_{\bar{K}_w \to 0} \Psi(\bar{K}_w) = -\frac{2\bar{k}_c f_i}{\bar{k}_i - \bar{k}_c}(< 0). \tag{A.3}$$

Since the level of Ψ monotonically increases with $\bar{K}_w$, we obtain the uniquely determined value of aggregate capital in the steady state. Moreover, when the steady-state value of capital, $\bar{K}_w$, is substituted into $(A.1)$, we confirm that the steady-state value of aggregate consumption is uniquely determined.

As for the stability analysis of steady state, the determinant of our system in (19) and (26) is given by:

$$\begin{aligned} \text{Det} &= \frac{\partial \dot{C}_w}{\partial C_w}\frac{\partial \dot{K}_w}{\partial K_w} - \frac{\partial \dot{C}_w}{\partial K_w}\frac{\partial \dot{K}_w}{\partial C_w}, \\ &= \frac{\bar{C}_w(\bar{\Omega} + \bar{\Omega}^*)}{\bar{Z}_{hf}\bar{C}_w + \bar{\Omega} + \bar{\Omega}^*}\frac{r'\bar{p}}{\bar{C}_w}\left(\frac{f_i}{\bar{k}_i - \bar{k}_c} - \delta\right), \end{aligned} \tag{A.4}$$

[8] As for $\Psi'(\bar{K}_w) > 0$, using $\dot{K}_w = 0$, we can show that

$$\frac{f_i}{\bar{k}_i - \bar{k}_c} - \delta = \frac{f_i}{\bar{k}_i - \bar{k}_c} - \frac{G(\bar{K}_w, \bar{C}_w)}{\bar{K}_w} = \frac{2 f_i \bar{k}_c}{\bar{K}_w(\bar{k}_i - \bar{k}_c)} > 0.$$

where we make use of

$$\frac{\partial \dot{C}_w}{\partial C_w} = \frac{\bar{Z}_{hf}\bar{C}_w(\bar{\Omega}+\bar{\Omega}^*)}{\bar{Z}_{hf}\bar{C}_w+\bar{\Omega}+\bar{\Omega}^*}\left(r'\frac{\partial p}{\partial C_w} - \frac{G_{C_w}}{\bar{Z}(2\bar{k}_i - \bar{K}_w)}\right) < 0,$$

$$\frac{\partial \dot{C}_w}{\partial K_w} = \frac{\bar{Z}_{hf}\bar{C}_w(\bar{\Omega}+\bar{\Omega}^*)}{\bar{Z}_{hf}\bar{C}_w+\bar{\Omega}+\bar{\Omega}^*} \times \frac{r'\bar{p} - G_{K_w} + \delta}{\bar{Z}_{hf}(2\bar{k}_i - \bar{K}_w)}$$

$$\frac{\partial \dot{K}_w}{\partial C_w} = G_{C_w} < 0$$

$$\frac{\partial \dot{K}_w}{\partial K_w} = G_{K_w} - \delta$$

Using the condition (29), we can lead to the negative sign of determinant in $(A.4)$, which means that the unique steady-state equilibrium is saddle-path stable.

Appendix B

To derive $(36b)$, we firstly see the following:

$$-\frac{\partial p}{\partial C_w}G_{K_w} + \frac{\partial p}{\partial K_w}G_{C_w} = -\frac{\partial p}{\partial C_w}\frac{f_i}{\bar{k}_i - \bar{k}_c}. \quad \text{(B.1)}$$

Making use of $(B.1)$ and $\frac{f_i}{\bar{k}_i - \bar{k}_c} - \delta = \frac{2\bar{k}_c f_i}{\bar{K}_w(\bar{k}_i - \bar{k}_c)}$, we can give $(36a)$ as follows:

$$\bar{D}_{hf} = \frac{2\bar{p}}{G_{C_w}}\Bigg\{\underbrace{\lambda\left(\bar{k}_c f_c''\bar{k}_c'\frac{\partial p}{\partial C_w} + \frac{\bar{w}}{\bar{p}\bar{C}_w}\right)}_{(\#B1)}\underbrace{-\frac{2\bar{k}_c f_i}{\bar{K}_w(\bar{k}_i - \bar{k}_c)}\left(\bar{k}_c f_c''\bar{k}_c'\frac{\partial p}{\partial C_w} + \frac{\bar{w}}{\bar{p}\bar{C}_w}\right)}_{(\#B2)}$$
$$\underbrace{-\frac{\bar{w}}{\bar{p}\bar{C}_w}\frac{f_i\frac{\partial p}{\partial K_w}}{(\bar{k}_i - \bar{k}_c)^2}\left[\bar{k}_c'(\bar{K}_w - 2\bar{k}_i) + \bar{k}_i'(\bar{K}_w - 2\bar{k}_c)\left(\frac{(\bar{k}_i - \bar{k}_c)f_i'}{f_i} - 1\right)\right]}_{(\#B3)}\Bigg\}. \quad \text{(B.2)}$$

Using $f_c(k_c) = k_c^{\alpha_c}$, we can show that $(\#B1)$ has a positive sign:

$$(\#B1) = \frac{\bar{w}}{\bar{Z}_{hf}\bar{C}_w(\bar{k}_i - \bar{k}_c)}\left(\frac{\bar{k}_i'(\bar{K}_w - 2\bar{k}_c)}{2\bar{k}_i - \bar{K}_w} + \bar{k}_c'\right)(> 0). \quad \text{(B.3)}$$

The sum of rest terms $(\#B2)$ and $(\#B3)$ is arranged as follows:

$$(\#B2) + (\#B3) = -\frac{f_i}{(\bar{k}_i - \bar{k}_c)\bar{Z}_{hf}}\Bigg\{\frac{2\bar{k}_c}{\bar{K}_w\bar{C}_w}\left[\underbrace{\bar{k}_c f_c''\bar{k}_c'\bar{p}}_{(\#B4)} + \frac{\bar{w}}{\bar{p}}\left(\underbrace{\frac{f_c'\bar{p}\bar{k}_c'}{f_c}}_{(\#B5)} + \underbrace{\frac{\bar{p}(\bar{K}_w - 2\bar{k}_c)\bar{k}_i'}{(2\bar{k}_i - \bar{K}_w)(\bar{k}_i - \bar{k}_c)}}_{(\#B6)} + \underbrace{\frac{\bar{k}_c'\bar{p}}{\bar{k}_i - \bar{k}_c}}_{(\#B7)}\right)\right]$$
$$+\frac{\bar{w}}{\bar{C}_w(2\bar{k}_i - \bar{K}_w)(\bar{k}_i - \bar{k}_c)}\left(\underbrace{\bar{k}_c'(\bar{K}_w - 2\bar{k}_i)}_{(\#B8)} + \underbrace{\bar{k}_i'(\bar{K}_w - 2\bar{k}_c)\left(\frac{(\bar{k}_i - \bar{k}_c)f_i'}{f_i} - 1\right)}_{(\#B9)}\right)\Bigg\} \quad \text{(B.4)}$$

As for each term, we can obtain the following:

$$(\#B6)+(\#B9)=\frac{\bar{w}(\bar{K}_w-2\bar{k}_c)\bar{k}_i'}{\bar{C}_w(2\bar{k}_i-\bar{K}_w)(\bar{k}_i-\bar{k}_c)}\left(\frac{2\bar{k}_c}{\bar{K}_w}+\frac{(\bar{k}_i-\bar{k}_c)f_i'}{f_i}-1\right), \tag{B.5a}$$

$$(\#B4)+(\#B5)+(\#B7)+(\#B8)=\frac{\bar{k}_c'\bar{w}}{\bar{C}_w(\bar{k}_i-\bar{k}_c)}\left[\frac{2(\bar{k}_i-\bar{k}_c)\bar{k}_c f_c'}{\bar{K}_w(f_c-\bar{k}_c f_c')}\underbrace{\left(\frac{\bar{k}_c f_c''}{f_c'}+1-\frac{\bar{k}_c f_c'}{f_c}\right)}_{(\#B9)}+\frac{2\bar{k}_c}{\bar{K}_w}-1\right]. \tag{B.5b}$$

When the production function is specified by the Cobb-Douglas type, $(\#B9)$ is zero. Then, under the Cobb-Douglas type of production function, the sum of $(B.5a)$ and $(B.5b)$ is given by

$$(\#B4)+(\#B5)+(\#B6)+(\#B7)+(\#B8)+(\#B9)=-\frac{\bar{w}(\bar{K}_w-2\bar{k}_c)\bar{k}_i'(1-\alpha_i)}{\bar{k}_i\bar{C}_w(2\bar{k}_i-\bar{K}_w)}(>0). \tag{B.6}$$

Finally, we can confirm the sum of $(\#B2)$ and $(\#B3)$ as follows:

$$(\#B2)+(\#B3)=\frac{f_i\bar{w}(\bar{K}_w-2\bar{k}_c)\bar{k}_i'(1-\alpha_i)}{\bar{k}_i\bar{Z}_{hf}\bar{C}_w(\bar{k}_i-\bar{k}_c)(2\bar{k}_i-\bar{K}_w)}(>0). \tag{B.7}$$

Next, from $(B.1)$ and $(B.7)$ we can show that

$$\begin{aligned}\bar{D}_{hf}&=\frac{2\bar{p}}{G_{C_w}}\left[\frac{\lambda\bar{w}}{\bar{Z}_{hf}\bar{C}_w(\bar{k}_i-\bar{k}_c)}\left(\frac{\bar{k}_i'(\bar{K}_w-2\bar{k}_c)}{2\bar{k}_i-\bar{K}_w}+\bar{k}_c'\right)+\frac{f_i\bar{w}(\bar{K}_w-2\bar{k}_c)\bar{k}_i'(1-\alpha_i)}{\bar{k}_i\bar{Z}_{hf}\bar{C}_w(\bar{k}_i-\bar{k}_c)(2\bar{k}_i-\bar{K}_w)}\right],\\&=\frac{2\bar{p}\bar{w}\bar{k}_i'}{G_{C_w}\bar{Z}_{hf}\bar{C}_w(\bar{k}_i-\bar{k}_c)}\left[\lambda\left(\frac{\bar{K}_w-2\bar{k}_c}{2\bar{k}_i-\bar{K}_w}+\frac{\bar{k}_c'}{\bar{k}_i'}\right)+\frac{(1-\alpha_i)f_i(\bar{K}_w-2\bar{k}_c)}{\bar{k}_i(2\bar{k}_i-\bar{K}_w)}\right].\end{aligned} \tag{B.8}$$

We note that the Cobb-Douglas type of production function gives the following:

$$\frac{\bar{k}_c'}{\bar{k}_i'}=\frac{\alpha_c(1-\alpha_i)}{\alpha_i(1-\alpha_c)},\qquad \frac{f_i}{\bar{k}_i}=\frac{\delta+\rho}{\alpha_i}, \tag{B.9}$$

and hence, $(B.10)$ is modified by

$$\bar{D}_{hf}=\frac{2\bar{p}\bar{w}\bar{k}_i'}{G_{C_w}\bar{Z}_{hf}\bar{C}_w(\bar{k}_i-\bar{k}_c)}\left[\lambda\left(\frac{\bar{K}_w-2\bar{k}_c}{2\bar{k}_i-\bar{K}_w}+\frac{\alpha_c(1-\alpha_i)}{\alpha_i(1-\alpha_c)}\right)+\frac{(1-\alpha_i)(\delta+\rho)(\bar{K}_w-2\bar{k}_c)}{\alpha_i(2\bar{k}_i-\bar{K}_w)}\right]. \tag{B.10}$$

Turning our interests into $\left(\frac{\bar{K}_w-2\bar{k}_c}{2\bar{k}_i-\bar{K}_w}\right)$, from the specification of production function we can show that

$$\frac{\bar{K}_w-2\bar{k}_c}{2\bar{k}_i-\bar{K}_w}=\frac{\frac{\bar{K}_w}{\bar{k}_c}-2}{2\frac{\bar{k}_i}{\bar{k}_c}-\frac{\bar{K}_w}{\bar{k}_c}}, \tag{B.11}$$

where from $(1a)$ and $(1b)$, we note that $\bar{k}_i/\bar{k}_c$ is given by the production parameters:

$$\frac{\bar{k}_i}{\bar{k}_c} = \frac{\alpha_i(1-\alpha_c)}{\alpha_c(1-\alpha_i)}. \tag{B.12}$$

As for $\bar{K}_w/\bar{k}_c$, we make use of $(28a)$ as follows:

$$\frac{\bar{K}_w - 2\bar{k}_c}{\bar{k}_i - \bar{k}_c} f_i = \delta \bar{K}_w, \quad \Leftrightarrow \quad \frac{\bar{K}_w}{\bar{k}_c} = \frac{1}{\frac{1}{2} - \frac{\delta \bar{k}_i(1-\bar{k}_c/\bar{k}_i)}{2f_i}}, \tag{B.13}$$

which leads to the following:

$$\frac{\bar{K}_w}{\bar{k}_c} = \frac{2(\delta+\rho)}{\delta + \rho - \alpha_i\delta\left(1 - \frac{\alpha_c(1-\alpha_i)}{\alpha_i(1-\alpha_c)}\right)}. \tag{B.14}$$

Finally, substituting $(B.14)$ into $(B.11)$, we can show that

$$\frac{\bar{K}_w - 2\bar{k}_c}{2\bar{k}_i - \bar{K}_w} = \frac{\alpha_c(1-\alpha_i)}{\alpha_i(1-\alpha_c)} \times \frac{\alpha_i\delta}{\delta+\rho-\alpha_i\delta}(>0). \tag{B.15}$$

Then, substituting $(B.15)$ into $(B.10)$, we can show:

$$\bar{D}_{hf} = \frac{2\bar{p}\bar{w}\bar{k}_i'}{G_{C_w}\bar{Z}_{hf}\bar{C}_w(\bar{k}_i - \bar{k}_c)} \times \frac{\alpha_c(1-\alpha_i)(\delta+\rho)}{\alpha_i(1-\alpha_c)(\delta+\rho-\alpha_i\delta)}\left[\lambda + \alpha_i(1-\alpha_i)\delta\right]. \tag{B.16}$$

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