Macroeconomic Dynamics of Human Development:
Indeterminacy and Bifurcation due to
Productive-Consumption Externality

Ichiroh Daitoh†
Keio University

August 30, 2014

This paper investigates macroeconomic dynamics of underdeveloped societies without enough nutrition, health and/or basic education. We introduce the "productive consumption (PC) hypothesis", whose idea is essentially the same as the efficiency wage hypothesis by Leibenstein (1957), into a standard RBC model with factor-generated externalities. First, indeterminacy happens due to the PC externality in the absence of capital externality. Second, in contrast to the standard growth models, the "intertemporal *complementarity* between present and future consumption" may hold under a saddle-point stable steady state equilibrium (SE). Third, under indeterminacy, the PC effect contributes to generate cyclical equilibrium paths. When the *super*critical Hoph bifurcation occurs, equilibrium paths may either diverge from an SE point or exhibit an endogenous cycle. These non-monotonic behaviors are consistent with empirical data. Forth, when the *sub*critical Hoph bifurcation occurs, even when the initial capital falls short of a critical level, we can avoid the "corridor stability" if coordination of expectation is possible. If it is impossible, an economy fluctuates permanently with capital periodically approaching to zero. When no equilibrium paths exist, it could be interpreted as a failure to establish a competitive market economy system. The "Big Push" may help escape from this new type of "underdevelopment trap". Finally, an introduction of human development aid can induce a cyclical path converging to a new SE with higher welfare. We cannot always expect a underdeveloped society to follow a monotonic growth path even if the aid improves SE welfare.

Keywords: Hoph Bifurcation, Human Development, Indeterminacy of Equilibrium, Productive Consumption Hypothesis,

JEL Classification Codes：O11, O15, E13

---

† Faculty of Business and Commerce, Keio University: 2-15-45, Minato-ku, Tokyo, 108-8345, Japan; Tel & Fax: +81-3-5418-6707, E-mail: idaitoh@fbc.keio.ac.jp

1. Introduction

Since the beginning of this century the Millennium Development Goals (MDGs) have been playing a central role in the world-wide development strategies for modern underdeveloped countries. Economists and international institutions such as the World Bank have been emphasizing the role of “poverty reduction” strategies through an improvement of nutrition, health, sanitation and/or basic education (see Table 1). These strategies,[1] at least implicitly, presume an economic growth mechanism in which an improvement of these utility-increasing factors can improve workers’ productivity and thus increase saving and investment, accelerating capital accumulation. However, no growth theories that explicitly formulate this economic mechanism have been presented so far. Even in the important discussions about whether foreign aid will play a useful or harmful role to promote the poverty reduction in underdeveloped societies (Sachs;2005, Easterly;2006), economists have never provided a theoretical foundation for their views by clarifying properties of the equilibrium paths derived from this growth mechanism.

The importance of nutrition, health, sanitation and/or basic education for growth of underdeveloped economies has first been emphasized by Leibenstein (1957a,b) in the form of the “efficiency wage hypothesis (EWH)”. For the last several decades this hypothesis has been extensively studied in development economics and macroeconomics. Nevertheless, we have not fully known consequences of this economic mechanism and properties of growth paths of the entire economy.

How does this mechanism differ from those in the conventional growth models like Solow-type neoclassical or Ramsey-Cass-Koopmans-type optimal growth models? In

[1] The poverty reduction strategies are based on the concept of “Human Development (HD)”, which primarily means an enhancement of human rights due to an improvement of these utility-increasing activities. Since the World Bank’s “structural adjustments” strategy in the 1980s turned out to increase poverty world-wide, this concept has gained importance in development strategies.

the conventional models, the main engine of growth is *investment* and thus growth is accelerated by *decreasing* present consumption from a given amount of income. By contrast, in this growth mechanism with an improvement of the consumption (utility-increasing) activities, not a decrease but an *increase* in present consumption improves workers' productivity and thus tends to increase production and investment. This feature can induce qualitatively different aspects into the growth process of poor economies.

This paper investigates macroeconomic dynamics of underdeveloped societies without enough nutrition, health and/or basic education. In order to formulate a tractable and general framework, we introduce the idea of the EWH into a standard real business cycle (RBC) model with factor-generated externalities (Benhabib and Farmer (1994)). The idea of the EWH is formulated in the form of the "productive consumption hypothesis (PCH)" by Steger (2000, 2002), in which an increase in per capita consumption, not in wage, raises workers' productivity.

It will be useful for readers if we explain the strands of research from the EWH to the PCH in development economics and macroeconomics. The EWH was analyzed in static models in 1970s-80s, focusing on rural labor markets in developing economies (Stiglitz, 1976; Bliss and Stern, 1978; Gersovitz, 1983; Dasgupta and Ray, 1986). In 1990s, it was often used as one of the important theories explaining wage rigidity and involuntary unemployment in Keynesian macroeconomics. Interesting results such as the existence of multiple equilibria were derived based on dynamic models. (Ray and Streufert, 1993; Banerji and Gupta, 1997; Jellala and Zenoub, 2000). These studies, however, tended to focus on the issues *within labor market*.

Steger (2000, 2002) presented two new formulations of the idea of the EWH which enable us to analyze a *growth process of the entire economy*, going beyond labor market issues. To emphasize this point he renamed it as the "productive consumption hypothesis (PCH)". The first formulation was the one in which an increase in per capita consumption accelerates (disembodied) human capital accumulation. The second formulation was the one in which an increase in per capita consumption raises workers' productivity at a point in time (as in the EWH). Gupta (2003) proposed another formulation in which some part of per capita consumption raises only the level of human capital and the other only utility. Unfortunately, all these models had either no balanced growth path (BGP) as a steady-state equilibrium (SE) (Steger;2000, 2002) or only a perfectly *un*stable SE (Gupta;2003). Because of this deficiency, economists had not been able to use the standard methods of growth analysis such as phase diagram. Nor had we been able to engage in comparative statics and dynamics of the SE. These difficulties have been overcome by using Steger's first formulation of the PCH by Daitoh (2010). Using these standard methods, he explains the population dynamics frequently observed in underdeveloped economies by considering a regime shift from a zero-saving to a positive-saving phase. In this paper, we explore consequences and implications of the productive-consumption (PC) dynamics under Steger's second formulation of the PCH.

We have five main conclusions. First, indeterminacy happens due to the PC externality even in the absence of capital externality, implying that our model could be appropriate for capital-scarce underdeveloped economies. Second, in contrast to the standard growth models, the "intertemporal *complementarity* between present and future consumption" may hold under a saddle-point stable SE. Third, under indeterminacy, the PC effect contributes to generate cyclical behaviors of equilibrium paths converging to an SE. When the *super*critical Hoph bifurcation occurs, equilibrium paths may either diverge from an SE point or exhibit an endogenous cycle. These non-monotonic behaviors are consistent with empirical data. Forth, when the *sub*critical Hoph bifurcation occurs, even when initial capital stock falls short of a critical level, we can avoid the "corridor stability" if coordination of expectation is possible. If it is impossible, capital stock fluctuates permanently, periodically approaching to zero. When all the paths do not satisfy transversality condition, no competitive equilibrium paths exist, meaning a failure to establish a competitive market economy system. The "Big Push" may help escape from this new type of "underdevelopment trap". Finally, an introduction of human development aid can induce a cyclical path converging to a new SE with higher welfare. We cannot always expect an underdeveloped society to follow a monotonic growth path even if the aid improves SE welfare.

2. The Model

We construct a one-sector RBC model with factor-generated externalities with the PCH. Throughout the paper, total population is assumed to be constant over time and normalized to unity. We assume that the aggregate (per capita) production function $y = f(k, h(c)n; \bar{X})$ exhibits *constant* returns to scale in capital k and efficiency labor $h(c)n$, where y is real GDP, n the number of working hours that he/she supplies and $h(c)$ his labor productivity per working hour. To clarify the analyses, we specify

$$f(k, h(c)n; \bar{X}) = k^a [h(c)n]^{1-a} \bar{X} \qquad 0 < a < 1 \tag{1}$$

The term $\bar{X} = X(\bar{k}, h(\bar{c})\bar{n})$ represents positive externalities of capital and labor to individual firms, where variables with an upper bar $\bar{k}$, $\bar{n}$ and $\bar{c}$ are as expected values. From a social point of view, the per capita production function exhibits *increasing* returns to scale. Thus we assume

$$\bar{X} = \bar{k}^{\alpha - a}[h(\bar{c})\bar{n}]^{\beta-(1-a)} \qquad \alpha \geq a\,, \quad \beta \geq 1-a \tag{2}$$

Because $\alpha + \beta > 1$ holds, the social production function $f(k, h(c)n) = k^{\alpha}[h(c)n]^{\beta}$ certainly exhibits *increasing* returns to scale. Note that $\alpha = a$ holds in the absence of capital externality.

The PCH is introduced in the form of $h(c)$ function with $h'(c) > 0$: an increase in (per capita) consumption c raises each worker's labor productivity $h(c)$.[2] Thus $h(c)$ function will be called "Productive Consumption (PC) function", whose idea is essentially the same as that in the "efficiency wage hypothesis". Furthermore, we assume that $h(c)$ is external to individuals because people in a poor society cannot fully control how much their productivity rises by an increase in consumption (even if they can recognize it): in a society with poor social or health infrastructure, people cannot help drinking unsafe water for survival, or have medical services (e.g., injection) under poor sanitation with high risks of diseases (e.g., HIV/AIDS or hepatitis). When average consumption in such a society increases, people will be able to improve their nutrition (physical strength or immunity), health (quality of medical services) and basic education (access to knowledge useful to protect themselves against diseases). This improves labor productivity of an individual worker.[3] In this sense, per capita consumption in the PC function $h(c)$ could be regarded as the average consumption in a society and thus as the expected value $\bar{c}$ to an individual worker. Therefore, we will

---

[2] This corresponds to the second formulation for the PCH proposed by Steger (2002),

[3] You could interpret it as the concept of "Marshallian externalities" being applied to the consumption side.

treat all $h(c)$ terms as $h(\overline{c})$ in the following analysis for a decentralized market economy.

We introduce the key parameter ε of this paper by specifying the PC function as

$$h(c) = c^{\varepsilon}, \qquad \varepsilon > 0. \tag{3}$$

It is concave for $0 < \varepsilon < 1$ while it is convex for $\varepsilon > 1$. Taking $ch''(c)/h'(c) = \varepsilon - 1$ into consideration, when a value of ε is close to one the graph of $h(c)$ is close to linear (i.e., has a small curvature). The reason why we assume this simple function (with no inflection points) instead of the S-shaped curve in the efficiency wage models is that it must be no less interesting to find the possibility of complicated dynamics based on a simpler function.[4]

2.1 Equilibrium of Decentralized Market Economy

This economy consists of many identical consumers whose instantaneous utility function $u(c_t, l_t)$ is assumed to be concave in consumption c_t and leisure l_t ($u_{cc} < 0$ and $u_{cc}u_{ll} - u_{cl}u_{lc} > 0$). By specifying $u(c_t, l_t)$, the intertemporal utility is;

$$U = \int_0^{\infty} \left[\log c_t - \frac{(1-l_t)^{1+\chi}}{1+\chi} \right] e^{-\rho t} dt \qquad \chi \geq 0, \qquad \rho > 0 \tag{4}.$$

Consumers own all the capital and each worker has one unit of time which can be allocated between leisure l_t and working time n_t. Thus the time constraint is

$$n_t + l_t = 1. \tag{5}$$

---

4 If we assumed a U-shaped PC function, we could obtain multiple steady states and thus discuss the selection between high- and low-level equilibria (with Pareto-ranking). However, since the beginning of 1990s this kind of analysis has already been done very frequently (e.g., Becker *et al*, 1990; Matsuyama, 1991). We would like to proceed in a new and different direction which we consider no less important to understand development processes of poor countries.

Given $\bar{X}$, we will consider perfect foresight equilibrium. The rental rate r of capital and the wage rate w for one unit of efficiency labor are equal to the marginal product of capital and labor, respectively:

$$r = \frac{\partial f(k, h(\bar{c})n; \bar{X})}{\partial k} = ak^{\alpha-1}n^{\beta}c^{\varepsilon\beta}, \quad w = \frac{\partial f(k, h(\bar{c})n; \bar{X})}{\partial [h(\bar{c})n]} = (1-a)k^{\alpha}n^{\beta-1}c^{\varepsilon(\beta-1)} \tag{6}$$

The aggregate flow budget constraint is $r_t K_t + w_t N_t = C_t + \dot{K}_t - \delta K_t$, which leads to

$$\dot{k}_t = (r_t - \delta)k_t + w_t h(\bar{c}_t)n_t - c_t \qquad (7).$$

In a decentralized market economy, given the expectations of time paths $\{w_t\}_{t=0}^{\infty}$, $\{r_t\}_{t=0}^{\infty}$ and $\{h(\bar{c}_t)\}_{t=0}^{\infty}$, the representative consumer chooses time paths $\{c_t\}_{t=0}^{\infty}$ and $\{n_t\}_{t=0}^{\infty}$ to maximize (4) subject to the time constraint (5) and the flow budget constraint (7). The current-value Hamiltonian function is defined as:

$$H(c_t, n_t, k_t, p_t) = \log c_t - n^{1+\chi}/(1+\chi) + p_t[(r_t - \delta)k_t + w_t h(\bar{c})n_t - c_t] \tag{8}$$

where p_t is a costate variable. Under $\bar{k} = k$, $\bar{n} = n$ and $\bar{c} = c$, the first-order necessary condition (FOC) leads to

$$\frac{1}{c} = p \tag{10}$$

$$n^{\chi} = p(1-a)k^{\alpha}n^{\beta-1}c^{\varepsilon\beta} \tag{11}$$

$$\dot{p} = p[\rho + \delta - ak^{\alpha-1}n^{\beta}c^{\varepsilon\beta}] \tag{12}$$

and the transversality condition (TVC) $\lim_{t\to\infty} k_t p_t e^{-\rho t} = 0$. From (10) c and p change in the opposite direction while from (6) c and r in the same direction. Notice here that the second term on the right-hand side of (8) is linear in c and n from the consumer's viewpoint, because he regards $h(\bar{c})$ as given. Since $u(c, 1-n)$ is concave, (8) is concave in c, n and k. Thus the paths satisfying the FOCs are certainly equilibrium paths.

Dividing (11) by (10), we obtain the labor market equilibrium condition:

$$c^{1-\varepsilon}n^{\chi} = (1-a)k^{\alpha}n^{\beta-1}c^{\varepsilon(\beta-1)} \tag{13}$$

Given c, the left-hand side represents the labor supply curve while the right-hand side the labor demand curve. Using (10) and (13), n can be represented as a function of k and p:

$$n = \left[(1-a)k^{\alpha}p^{1-\varepsilon\beta}\right]^{\frac{1}{1+\chi-\beta}} \tag{14}$$

The equilibrium system of a decentralized market economy is represented as a two-dimensional dynamical system of k and p:

$$\dot{k}_t = Ak_t^{\Delta}p_t^{\Omega} - \left(\frac{1}{p_t}\right) - \delta k_t = K(k_t, p_t) \tag{15}$$

$$\dot{p}_t = p_t\left[\rho + \delta - aAk_t^{\Delta-1}p_t^{\Omega}\right] = P(k_t, p_t) \tag{16}$$

where

$$\Delta = \frac{\alpha(1+\chi)}{1+\chi-\beta} \qquad \Omega = \frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta} \qquad \text{with} \quad A = (1-a)^{\frac{\beta}{1+\chi-\beta}}.$$

2.2 Steady-state Equilibrium and its Stability

The steady-state equilibrium (SE) is defined by $\dot{k} = \dot{p} = 0$. The SE values (k^*, p^*) thus satisfy

$$A(k^*)^{\Delta}(p^*)^{\Omega} = \frac{1}{p^*} + \delta k^* \tag{17}$$

$$A(k^*)^{\Delta}(p^*)^{\Omega} = \left(\frac{\rho+\delta}{a}\right)k^* \tag{18}$$

We call the locus of (k, p) that satisfies (17) and (18) the *kk* curve and the *pp* curve, respectively. The slope of the *kk* curve and that of the *pp* curve in the neighborhood of an SE are, respectively,

$$\frac{dp}{dk} = -\frac{K_k^*}{K_p^*} = \frac{\delta - \Delta A k^{\Delta-1} p^{\Omega}}{A k^{\Delta}\Omega p^{\Omega-1} + (1/p^2)} \quad , \qquad \frac{dp}{dk} = -\frac{P_k^*}{P_p^*} = -\frac{(\Delta-1)p}{k\Omega} \tag{19}$$

However, their signs are ambiguous. In order to examine the stability of an SE, we derive the linearized system of (15) and (16) evaluated at the SE:

$$\begin{pmatrix} \dot{k} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} K_k^* & K_p^* \\ P_k^* & P_p^* \end{pmatrix} \begin{pmatrix} k-k^* \\ p-p^* \end{pmatrix} \tag{20}$$

Denoting the coefficient matrix as J^*, each term of J^* is

$$K_k^* = \Delta A(k^*)^{\Delta-1}(p^*)^{\Omega} - \delta$$

$$= \frac{\alpha(1+\chi)}{1+\chi-\beta}\left(\frac{\rho+\delta}{a}\right) - \delta = \frac{\alpha(1+\chi)\rho + [(\alpha-a)(1+\chi)+a\beta]\delta}{a(1+\chi-\beta)}$$

$$K_p^* = \Omega A(k^*)^{\Delta}(p^*)^{\Omega-1} + \frac{1}{(p^*)^2} = \frac{1}{(p^*)^2}\left\{1 + \frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}\left[\frac{\rho+\delta}{\rho+(1-a)\delta}\right]\right\}$$

$$P_k^* = p^*\left[-aA(\Delta-1)(k^*)^{\Delta-2}(p^*)^{\Omega}\right] = p^*\left[1 - \frac{\alpha(1+\chi)}{1+\chi-\beta}\right]\left(\frac{\rho+\delta}{k^*}\right)$$

$$= (p^*)^2\left[1 - \frac{\alpha(1+\chi)}{1+\chi-\beta}\right]\frac{[\rho+(1-a)\delta](\rho+\delta)}{a}$$

$$P_p^* = p^*[-aA(k^*)^{\Delta-1}\Omega(p^*)^{\Omega-1}] = -(\rho+\delta)\frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}$$

The SE is saddle-point stable if and only if $Det.J^* < 0$ holds, while it is perfectly stable (indeterminate) if and only if both $Trace.J^* < 0$ and $Det.J^* > 0$ hold, where

$$Trace.J^* = K_k^* + P_p^*$$

$$= \left[\frac{(\alpha - a)(1+\chi)\delta + \rho[\alpha(1+\chi) - a\beta]}{a(1+\chi-\beta)}\right] + \frac{\varepsilon\beta(1+\chi)(\rho+\delta)}{1+\chi-\beta} \tag{21}$$

$$Det.J^* = K_k^* P_p^* - K_p^* P_k^*$$

$$= -\frac{(\rho+\delta)(1+\chi)[\rho+(1-a)\delta]}{a}\frac{1-\alpha-\varepsilon\beta}{1+\chi-\beta} \tag{22}$$

We have two important findings from the above expressions. First, an SE is unique if it exists. This is because the signs of these terms are unambiguously determined if a set of exogenous parameter values are given. If we assume there exist more than one SE, we should conclude that equilibria with different stability properties coexist. However, this leads to a contradiction.[5] Second, labor externality needs to be as strong as in BF (1994) (i.e., $1+\chi<\beta$) for an SE to be perfectly stable. To see this, suppose that $1+\chi>\beta$ holds. Then $Trace.J^*>0$ necessarily holds because $\alpha(1+\chi)-a\beta>0$ holds under $\alpha \geq a$.

In the present model, an SE can be either saddle-point stable or unstable for $1+\chi>\beta$, while it can be perfectly stable as well for $1+\chi<\beta$.[6]

Lemma 1: *In the present model under the PCH, (i) an SE is unique if it exists. (ii) The labor externality needs to be as strong as in BF (1994, i.e.,* $1+\chi<\beta$ *for an SE to be perfectly stable (i.e., indeterminacy of equilibrium).*

[5] The author appreciates Professor Takumi Naito for his comment about this point.
[6] This property of the model is the same as in Benhabib and Farmer (1994).

3. Equilibrium Dynamics under Saddle-point Stability

In this paper we will focus mainly on transition equilibrium paths (see Appendix D for the justification). This section examines equilibrium dynamics for a saddle-point stable SE. Let us first see when it happens. From (22), $Det.J^* = K_k^* P_p^* - K_p^* P_k^* < 0$ holds iff $1-\alpha-\varepsilon\beta > 0$, which is equivalent to $(1-\alpha)/\beta > \varepsilon$. Thus $1 > \alpha$ is necessary, which is satisfied when $\alpha = a$ holds. By $\alpha + \beta > 1$, $1 > \varepsilon$ is also necessary. That is, for a saddle-point stable SE, the PC function $h(c)$ needs to be concave. This implies that a saddle-point stable SE, if it exists, would in concave part of the S-shaped efficiency function.

We have found, after examining all cases, that properties of a unique transition path converging to an SE are the same among most cases both for $1+\chi > \beta$ and for $1+\chi < \beta$: *c* and *k* both increase along an equilibrium path starting from low initial capital stock k_0. However, we find a new and interesting case specific to the PCH under $1+\chi > \beta$ ($K_k^* > 0$ always holds): *c* is initially chosen at a *high* level and begins to *decrease* along a transition path when capital accumulation sets in. The representative case above can be shown in Fig. 2-1 for small values of ε ($K_p^* > 0$) under $1 > \varepsilon > 1/(1+\chi)$ and $1+\chi > \beta$.[7] The new finding is shown in Fig. 2-2 for large values of ε ($K_p^* < 0$) under $1 > \varepsilon > 1/(1+\chi)$ and $1+\chi > \beta$.[8] Note that the slopes of the *kk* and *pp* curves depend on the sign of K_p^*.

---

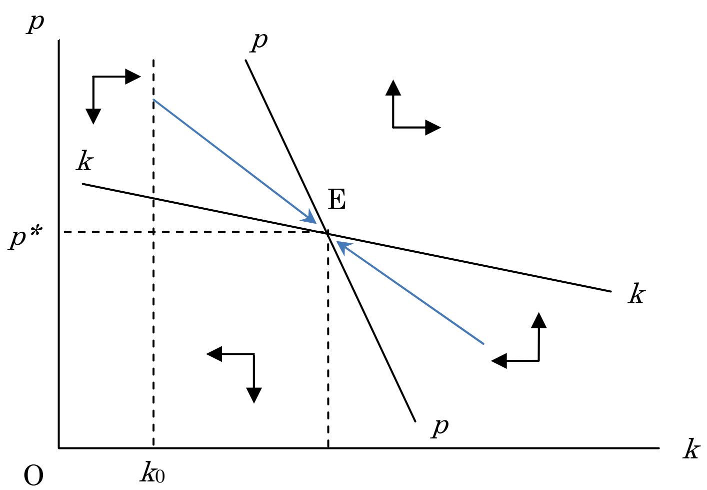
[7] We show the dynamics for $1 > \varepsilon(1+\lambda)$ in Fig. 1-1 and Fig.1-2 in Appendix, which are similar to this case.

[8] The PC parameter ε is included only in K_p^* and P_p^*. Suppose that the PC effect is absent ($\varepsilon = 0$). Then $K_p^* > 0$ and $P_p^* < 0$ hold for $1+\chi > \beta$ while $P_p^* > 0$ (the sign of K_p^* is ambiguous) holds for $1+\chi < \beta$.

3.1 Equilibrium Dynamics for $K_p^* > 0$

When the value of ε is so small that $K_p^* > 0$ holds under $1 > \varepsilon > 1/(1+\chi)$, $P_p^* > 0$ always holds while the sign P_k^* are ambiguous. With $K_p^* > 0$, $Det.J^* < 0$ leads to $0 > -(K_k^* / K_p^*) > -(P_k^* / P_p^*)$: the decreasing *kk* curve is flatter than the (decreasing) *pp* curve.[9] Fig. 2-1 shows a unique saddle-point path which starts from low initial capital stock k_0 and converges to the SE. Along this equilibrium path, consumption $c = 1/p$ is initially chosen at a low level and increases with capital k toward the SE.

Fig.2-1. Saddle-point stable SE for $1+\chi > \beta$, $1 > \varepsilon > 1/(1+\chi)$ and $K_p^* > 0$



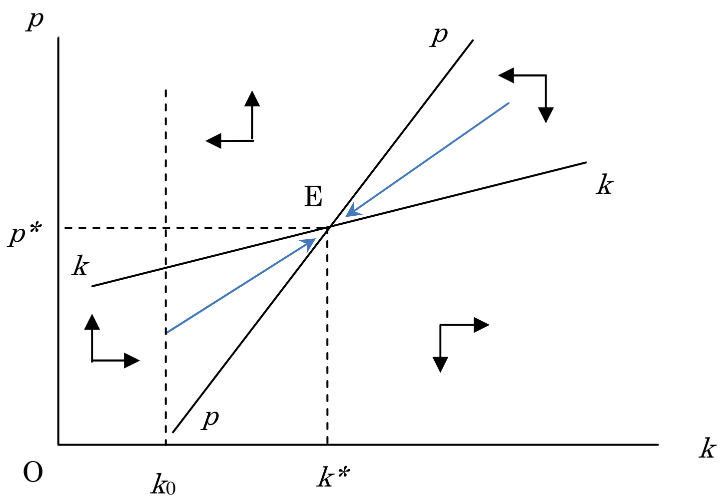
This property of equilibrium path is similar to that of Ramsey-Cass-Koopmans optimal growth model. However, an economic mechanism in our model has a different aspect because the PC effect is present. Let us explain it intuitively. In the case of $1+\chi > \beta$, the labor demand curve is typically decreasing ($\beta - 1 < 0$). A low initial consumption

---

9 In Appendix we confirm that each case of the figures is possible by examining whether the corresponding inequalities can hold simultaneously.

$c_0 = 1/p_0$ will make the labor demand curve $w = (1-a)k^{\alpha} n^{\beta-1} c^{\varepsilon(\beta-1)}$ lie in a high position but (because of $1-\varepsilon > 0$) the labor supply curve $w = c^{1-\varepsilon} n^{\chi}$ lie in a low position (see (13)). Both effects raise the equilibrium labor input n, and thus the labor input in efficiency units $h(c)n$ tends to be large. Therefore the rental rate r of capital will be high. By Euler equation (12), the costate variable p declines and thus c increases.

Fig. 2-2. Saddle-point stable SE for $1+\chi > \beta$, $1 > \varepsilon > 1/(1+\chi)$ and $K_p^* < 0$



3.2 Equilibrium Dynamics for $K_p^* < 0$

Let us proceed to the case of $K_p^* < 0$ (ε is close to 1) under $1 > \varepsilon > 1/(1+\chi)$. Then $Det.J^* < 0$ leads to $0 < -(K_k^* / K_p^*) < -(P_k^* / P_p^*)$, meaning that the increasing *kk* curve is flatter than the *pp* curve. Fig. 2-2 shows the equilibrium dynamics specific to the PCH: c is initially chosen at a *high* level c_0 and begins to *decrease* along the transition path starting from low initial capital stock k_0. An economic explanation for this equilibrium path can be made by reversing the logic for the case of $K_p^* > 0$.

We summarize the results for a saddle-point stable SE including those in Appendix.

Proposition 1: (Equilibrium Dynamics for Saddle-point Stable SE)

Suppose that an SE is saddle-point stable when labor externality is so weak that $1+\chi > \beta$ holds. Then both $1 > \alpha$ and $1 > \varepsilon$ necessarily hold.

(i) If the value of ε is so large that both $1 > \varepsilon > 1/(1+\chi)$ and $K_p^* < 0$ hold, consumption is initially chosen at a high level and then begins to decrease with capital accumulation along a transition path starting from a low level of initial capital stock.

(ii) In the other cases, both consumption and capital increase along a transition path starting from a low level of initial capital stock (as in the standard growth model like Ramsey-Cass-Koopmans' optimal growth model).

Suppose that an SE is saddle-point stable when $1+\chi < \beta$ holds, equilibrium dynamics has qualitatively the same properties as mentioned in (ii) above.

Let us elucidate why our PC growth model can generate this dynamics which differ from that in the standard growth model with the "intertemporal substitution (trade-off) between present and future consumption" (recall Ramsey-Cass-Koopmans optimal growth model). In our PC growth model, an increase in present consumption, because of an improvement in labor productivity ($h'(c_t) > 0$), leads to an increase in present production. This weakens the corresponding decrease in saving and investment, and therefore mitigates the harsh intertemporal trade-off between present and future consumption. This insight has already been provided by Steger (2000, 2002).

What is new in our PC growth analysis is to point out that if the PC effect is strong enough an increase in present consumption will not decrease but *increase* present investment.[10] Then future production and consumption tend to increase. This gives rise to the "intertemporal *complementarity* between present and future consumption". In such a situation, as shown in Fig. 2-2, consumers will initially choose high consumption expenditure and, by improving their labor productivity, can generate positive savings. Once capital begins to accumulate along the transition path, they can afford to decrease consumption, because an increase in capital k compensates the decrease in labor productivity. We have this special feature under $K_p^* < 0$, which means that an increase in c and thus a decline in $p = (1/c)$ increase the amount of saving K. Note that this would happen near the inflection point of the S-shaped efficiency function, because ε is close to 1.

4. Equilibrium Dynamics under Indeterminacy

This section investigates properties of equilibrium paths under indeterminacy. We do not need to worry about the fact that the necessary condition for indeterminacy $1+\chi < \beta$ was criticized as unrealistic in 1990s because it implies the labor demand curve has a steeper slope than the (increasing) Frisch labor supply curve. It has already been well known that indeterminacy can easily occur even under small externalities in

10 This does not always mean that consumers choose zero saving in our model, because they derive utility from a combination of consumption and leisure.

various realistic situations.[11] Therefore, we could concentrate on exploring properties of macroeconomic dynamics under indeterminacy without worrying about the fact that $1+\chi<\beta$ is by itself unrealistic.

Before deriving equilibrium paths, we should answer a theoretical question of whether this indeterminacy model can apply to underdeveloped economies with small or no capital externality. This is because both capital and labor externalities are needed for indeterminacy in the BF model.[12] As for capital externality, the level of capital stock at each point in time could be regarded as a proxy for useful knowledge that has been accumulated through past production experiences, as in the formulation of learning-by-doing effects by Arrow (1962). In this sense, capital has a positive external effect. Thus underdeveloped economies tend to have very small or no capital externality. This implies that indeterminacy cannot occur in those economies. However, in our model, indeterminacy can occur even without capital externality if the PC externality is strong enough. This is a new theoretical finding, implying that the BF model can apply not only to advanced countries but also to underdeveloped economies as a fundamental framework for macroeconomic dynamic analysis.

4.1 The PC Externality as a Source of Indeterminacy

Let us first clarify the role of capital externality for indeterminacy under $1+\chi<\beta$. Solving $Trace.J^*<0$ for α, we get

---

11 The reason why the parameter values for indeterminacy lie in unrealistic ranges in early studies was that they used too simple models. Extensions to more general models induce a possibility of indeterminacy easily: one-sector model (Benhabib and Nishimura,1998; Mino,2001), non-separable utility function (Mino,1999; Bennett and Farmer, 2000), and endogenizing capacity utilization (Wen,1998).

12 We can easily confirm it in our model with $\varepsilon=0$ by finding that $Trace.J^*>0$ holds when either $\alpha=a$ or $\beta=1-a$ holds.

$$\alpha > \frac{a(1+\chi)\delta + \beta[a\rho - \varepsilon(1+\chi)(\rho+\delta)]}{(1+\chi)(\rho+\delta)} = \underline{\alpha} \tag{23}$$

It means that capital externality α needs to be strong enough. Benhabib and Farmer (1994) have already pointed out this property.[13] A new finding of our paper is that the minimum value $\underline{\alpha}$ will be smaller when the value of the PC parameter ε is larger.

Even if the capital externality is absent, indeterminacy will happen when the PC externality is strong enough.[14] Setting $\alpha = a$ and $\beta > 1-a$, we get

$$Trace.J^* = \rho + \frac{\varepsilon\beta(1+\chi)(\rho+\delta)}{1+\chi-\beta} \tag{24}$$

Under $1+\chi < \beta$, $Trace.J^* < 0$ is equivalent to

$$\varepsilon_0 = \left(\frac{\rho}{\rho+\delta}\right)\left[\frac{1}{1+\chi} - \frac{1}{\beta}\right] < \varepsilon \tag{T}$$

where ε_0 is defined by $Trace.J^* = 0$. $Det.J^* > 0$ holds iff $1-\alpha-\varepsilon\beta > 0$, that is,

$$\varepsilon < \frac{1-a}{\beta} \tag{D}$$

Taking account of $(1-a)/\beta < 1/(1+\chi) < 1$, indeterminacy happens when the PC function is concave to the extent that $\varepsilon_0 < \varepsilon < (1-a)/\beta$ holds. This implies that indeterminacy would happen in concave part of the S-shaped efficiency function. For (T) and (D) to be compatible, it is sufficient that $\varepsilon_0 < (1-a)/\beta$ holds if the value of δ is so large that

$$\rho + \delta > \left(\frac{\rho}{1-a}\right)\left[\frac{\beta}{1+\chi} - 1\right] \tag{K}$$

---

13 Meng and Yip (2008) find that indeterminacy can arise when the felicity function is separable in consumption and leisure and there are negative capital externalities.

14 Benhabib and Farmer (1994) show that indeterminacy cannot occur in the absence of capital externality even when the labor externality is strong (p.29).

holds.[15] When $a=0.33$, $\rho=0.02$, $\delta=0.03$ ($\delta=0.1$) and $\chi=0$ hold, (K) requires $\beta<2.675$ ($\beta<5.02$). These constraints for β seem moderate because the upper bounds of β are well over one (recall that the necessary condition for indeterminacy with $\chi=0$ is $1<\beta$).

Proposition 2: (PC Externality as a Source of Indeterminacy)

Suppose that labor externality is so strong that $1+\chi<\beta$ holds. For indeterminacy to happen, capital externality α needs to be larger than $\underline{\alpha}$ defined by (23). Under these conditions, we find the following results.

(i) The range of α needed for indeterminacy is larger (the value of $\underline{\alpha}$ is lower) when *the value of ε is larger. That is, when the PC externality gets stronger, the possibility of indeterminacy will expand in such a direction that weaker capital externality is needed.*

(ii) Even when capital externality is absent ($\alpha=a$ and $\beta>1-a$), indeterminacy actually happens if the depreciation rate of capital δ is so large relative to a, β and χ that (K) holds and if the PC parameter ε satisfies both (T) and (D).

Let us make an intuitive explanation for why indeterminacy occurs due to the PC externality by considering the labor market equilibrium. First of all, $1+\chi<\beta$ means by (13) that the slope of the labor demand curve ($\beta-1$) exceeds that of the labor supply curve (χ). In order to show that an SE is perfectly stable (indeterminate), we should say that when the value of p deviates from the initial SE (k^*, p^*), it will return to its

---

15 Conversely, $Trace.J^*>0$ necessarily holds when $\alpha>a$ and $\beta=1-a$ hold. Therefore labor externality cannot be replaced with the PC externality.

initial value even if the SE value of k^* remains unchanged (i.e., we do not need to choose a new value of k from the saddle-point path). Let us now suppose that only the value of p increases from the SE (by some exogenous shock). Because c decreases, $h(c)$ declines due to the PC externality. It gives rise to a downward shift of the (upward sloping) labor demand curve. If the corresponding rise in n is large enough, efficiency labor $h(c)n$ also increases.[16] It raises the marginal productivity of capital, and thus the rental rate r of capital will increase. By Euler equation (12), $\dot{p}<0$ holds. Therefore, the value of p declines toward its initial value. Conversely, if the PC effect were absent in this process, the labor demand curve would not shift downward, and this mechanism would not work.

The explanation above also indicates why indeterminacy can occur without capital externality. The mechanism above is independent of (a movement of) the capital externality, because we have presumed that the value of k remains unchanged at the SE value k^*. In particular, even without capital externality, this logic keeps working when the PC externality is present under $1+\chi<\beta$.

4.2 Properties of Equilibrium Dynamics under Indeterminacy

We will now investigate properties of equilibrium dynamics under indeterminacy. They turn out to be very different from a unique monotonically converging path in the standard growth models (such as Ramsey-Cass-Koopmans optimal growth model) and have a cyclical behavior. Those properties must probably be the same as that derived from BF (1994) model, though they did not show them explicitly. Therefore the main

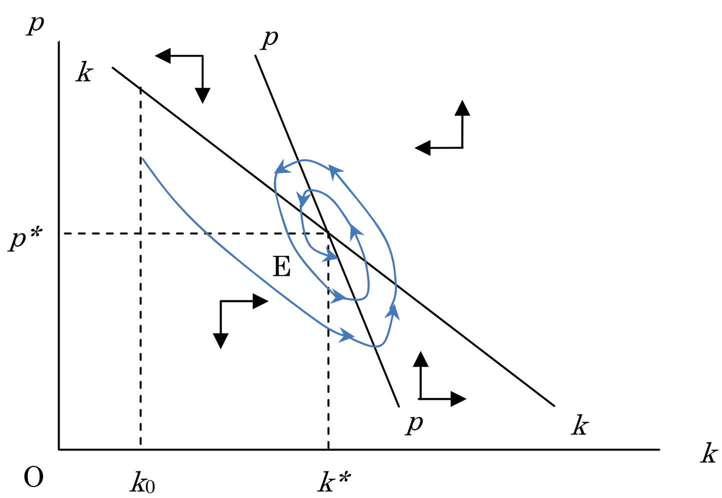
---

16 At the same time, the labor supply curve shifts down by the decrease in c. It tends to reduce n in equilibrium. Here we implicitly assume that the effect of a shift of the labor demand curve dominates. By emphasizing the labor demand side, we make it clear that this shift of the labor demand curve plays a key role for opening up the possibility of indeterminacy.

focus here should be on an economic mechanism of the PC effect generating this cyclical behavior, and on the realistic relevance of those properties.

First, let us show equilibrium paths diagrammatically. Under $1+\chi<\beta$, $K_k^*<0$ and $P_k^*>0$ holds. When $1<\varepsilon(1+\chi)$ holds, $1-\alpha-\varepsilon\beta<0$ and thus $Det.J^*<0$ must hold (recall condition (D)). Therefore indeterminacy cannot happen. When $1>\varepsilon(1+\chi)$ holds, $P_p^*>0$ holds (the sign of K_p^* is ambiguous). Thus $Det.J^*=K_k^*P_p^*-K_p^*P_k^*>0$ is equivalent to $-(P_k^*/P_p^*)<-(K_k^*/K_p^*)$: the (increasing or decreasing) *kk* curve has a larger slope than the decreasing *pp* curves. Fig.3-4 shows the case of $K_p^*<0$.

Fig.3-4. Equilibrium dynamics under Indeterminacy for $1+\chi<\beta$, $1>\varepsilon(1+\chi)$ and $K_p^*<0$



Under indeterminacy, as shown in Fig.3-4, a typical equilibrium path starting from low initial capital stock k_0 exhibits a damped oscillation with "overshooting" and “undershooting” around its SE level k^*, converging to the SE point.[17] In this case there exists a continuum of (transition) equilibrium paths starting from a given initial capital stock k_0. Then a realized path will be selected by “coordination of expectations” (e.g., Krugman;1991, Matsuyama;1991, Fukao and Benabou;1993, Lucas;1993)). Fig.3-4 illustrates one of the typical paths. You should note, however, that in underdeveloped societies with high diversity in non-economic (political, legal, social, cultural, or even religious) characteristics the social costs for coordinating expectations tend to be very large.

Proposition 3: (Equilibrium Dynamics under Indeterminacy)

Suppose that labor externality is as strong as in Benhabib and Farmer (1994), that is, $1+\chi<\beta$ holds, and that the SE is perfectly stable (indeterminate). Then, consumption and capital along a transition path typically exhibit a cyclical behavior, converging to an SE point. An equilibrium path will be chosen by coordination of expectations.

Now let us explain intuitively an economic mechanism of the PC effect generating this cyclical behavior. The relation between k_t and r_t along the marginal product of capital (MPK) curve and the relation between r_t and $c_t = 1/p_t$ through the Euler equation play a central role.

Starting from low initial capital k_0, the initial rental rate r_0 of capital is high and an incentive for capital accumulation is strong. The Euler equation requires a decline in p_t and thus an increase in c_t. When r_t decreases to the point with $\dot{p}_t = 0$, k_t continues

[17] You could consider two different types of equilibrium paths starting from the same initial capital stock k_0. One is a non-monotonic path along which capital decreases at first but begins to increase, converging to the SE with no cyclical movements. The other is a path that converges to the SE monotonically.

to increase. Thus r_t decreases further and c_t begins to decrease (p_t increases). The decrease in r_t weakens an incentive for capital accumulation.

The decrease in c_t, by the PC effect, leads to a decrease in $h(c_t)$. The reduction in efficiency labor $h(c_t)n_t$ tends to decrease production and therefore the saving and investment tend to decrease. When c_t decreases to the point with $\dot{k}_t = 0$, r_t begins to rise because of a reduction in k_t. This makes a decrease in c_t smaller by the Euler equation. After r_t reaches up to the point with $\dot{p}_t = 0$, c_t begins to increase (p_t decreases) again. This, by the PC effect, improves labor productivity $h(c_t)$ and the corresponding increase in $h(c_t)n_t$ leads to a rise in production and thus in saving and investment. When c_t reaches the point with $\dot{k}_t = 0$, capital k_t begins to accumulate again. The above explanation tells us a role of the PC effect in generating a cyclical behavior of an equilibrium path. This may contribute to a new understanding of growth process specific to underdeveloped economies under the PCH.

Finally, let us see whether these complicated behaviors of equilibrium paths could be consistent with reality. Since the independence of colonial areas in 1950s-60s, there have been low-income economies all over the world, especially in Asia, Latin America and Africa. Blue lines in “Table: Indeterminacy Cases” show the time series data (from *World Development Indicators*) of per capita real GDP in Latin American and African countries.[18] The red line in each table is the corresponding approximating curve. They represent the long-run trend, which could be interpreted as the SE path.[19] Then we find that per capita real GDP of Brazil, Guatemala, Algeria, Kenya and Swaziland first

---

[18] We use annual data of GDP because capital stock data are not available and consumption data lack in many years for low income and middle low income countries in WDI.

[19] However, the upward or downward (and horizontal) slope of the SE path (red line in the Tables) cannot be explained in the present model. They may be determined by, e.g., the availability of infrastructure, frequency of political or social conflicts, etc..

moved with ups and downs around the SE path, and their amplitudes have been damped gradually. These movements seem consistent with the oscillation in Fig.3-4. The data of per capita real GDP in Haiti during 1991-2011 may be consistent with a non-monotonic path without cyclical movements.

5. Equilibrium Dynamics under Hoph Bifurcation

This section investigates equilibrium dynamics when the Hoph bifurcation occurs. Let us start from the next lemma whose proof is given in Appendix.[20]

Lemma 2 *(Existence of Hoph Bifurcation):* With no capital externality ($\alpha = a$ and $\beta > 1 - a$), *an invariant closed curve (a periodic solution) surrounding the SE point emerges as a stationary equilibrium when the PC parameter ε changes. The bifurcation value is ε_0.*

The implications of the existence of a closed orbit as an equilibrium path will be different, depending on in which range of $\varepsilon < \varepsilon_0$ or $\varepsilon_0 < \varepsilon$ the closed orbit emerges. However, the condition for deciding which is more likely to happen generally depends on the third partial derivatives of $K(k, p)$ and $P(k, p)$ (Guckenheimer and Holmes (2002), p.152). Taking into consideration that we cannot generally tell the economic meaning of the third derivative of a function, we will not be able to say which situation is more likely to happen from an economic viewpoint. Thus we would like to show both

[20] We do not always have to examine how many invariant closed curves exist, because it would not add any new economic conclusions to the discussions below.

cases here without explicitly showing the condition for separating these situations, and explore their economic implications.[21]

5.1 Supercritical Hopf Bifurcation and Poverty Reduction by Human Development

If a closed orbit emerges for the small values of ε ($\varepsilon < \varepsilon_0$) which correspond to strongly concave PC functions, an SE point is perfectly unstable while the closed curve satisfies orbital stability. This is the case for the *supercritical* Hoph bifurcation. As shown in Fig.5-1, an equilibrium path will converge with cyclical movements to this closed orbit whether it starts from the initial point either within this closed curve or outside of it.[22] When an economy stays on this closed curve, the value of $k(t)p(t)$ is finite at any point in time and thus the TVC $\lim_{t\to\infty} k(t)p(t)e^{-\rho t} = 0$ is satisfied by the effect of the discount rate $\rho > 0$. Therefore this closed orbit will certainly be a stationary equilibrium in the long-run. Capital *k* and consumption *c* will not converge to the SE point but instead exhibit an endogenous periodic cycle.

Proposition 4: (Equilibrium Dynamics under Supercritical Hopf Bifurcation)

When a closed orbit emerges in the range of small values of the PC parameter ε ($\varepsilon < \varepsilon_0$), an SE point is unstable while the closed curve satisfies orbital stability.

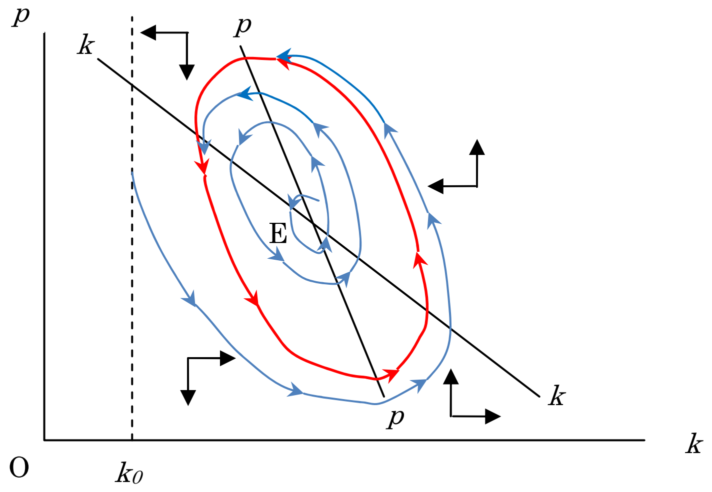
---

21 The similar way of presenting analytical results has been used in Benhabib and Miyao (1981) and Yoshida (2003).

22 The supercritical Hoph bifurcation happens when the *kk* and *pp* curves are both downward sloping with the *pp* curve steeper. This situation is the same as in Fig.3-4, where $1 > \varepsilon(1+\chi)$ and $K_p^* < 0$ hold. When $1 < \varepsilon(1+\chi)$ holds, the upward sloping *pp* curve must be flatter than the *kk* curve for $Det.J^* > 0$. However, $1 < \varepsilon(1+\chi)$ necessarily implies $Det.J^* < 0$, as explained in subsection 4.2. Therefore the supercritical Hoph bifurcation is shown by Fig.5-1.

Consumption and capital along an equilibrium path diverge from the SE point and/or converge to an endogenous periodic cycle.

Fig.5-1 Stable Endogenous Periodic Cycle (*super*critical Hopf bifurcation)



This possibility of an endogenous periodic cycle has already been pointed out in BF (1994). However, they have never examined its properties and implications. We have just mentioned its properties in Proposition 4 above.

These properties must have important implications to understand how the poverty reduction by human development (HD) will proceed. We should pay careful attention to the possibility that economies with the PC growth may not exhibit a (monotonic or cyclical) convergence toward an SE point (as shown in Fig. 2-1 or Fig.3-4). They may begin to diverge from the SE point and then keep moving on an invariant closed curve without approaching to the SE point. This result suggests it will be more difficult than has been expected to evaluate the success or failure of poverty reduction strategies including foreign aid.

Let us see whether these dynamics could be regarded as consistent with reality. "Tables: Bifurcation Cases" show that per capita real GDP of Latin American countries such as Belize, Panama and Puerto Rico moved along the long-run trend during the first decade of 1960-2011, and then diverged from it with expanding cyclical movements during the recent 30 years. We observe similar evidence for African countries such as Mali and Tunisia. These data seem consistent with a cyclical equilibrium path starting from a point within the closed orbit and converging to this orbit in Fig.5-1. In contrast, the per capita real GDP data of other African countries such as Chad, Democratic Republic of the Congo, the Gambia and Zambia show cyclical movements (i.e., ups and downs) around the SE path during the whole period. These movements seem consistent with the endogenous economic cycle in Fig.5-1.

5.2 Subcritical Hopf Bifurcation and New Implication of "Corridor Stability"

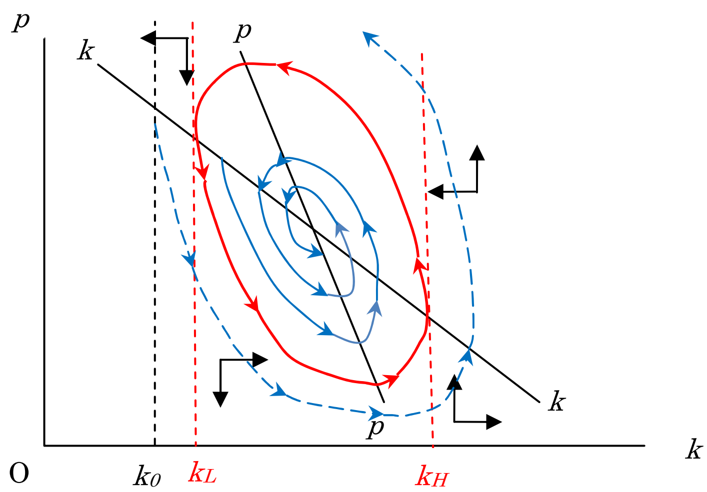
When a closed orbit emerges for large values of ε ($\varepsilon_0 < \varepsilon$) which corresponds to weakly concave PC functions, an SE point is perfectly stable while the closed curve is orbitally *un*stable. This is the case for the *subcritical* Hoph bifurcation.[23] Let us explain economic implications that are quite different from above by separating two possible types of equilibrium paths.

First, if a dynamic path starts from an initial point within this closed curve, as shown in Fig.5-2, it will converge cyclically to the SE point, satisfying the TVC. This is

[23] Fig.5-2 with the downward sloping *kk* and *pp* curves is appropriate for showing the subcritical Hoph bifurcation. First, under $1+\chi < \beta$, $Det.J^* > 0$ is equivalent to $1-\alpha-\varepsilon\beta > 0$. Then $1 > \varepsilon(1+\chi)$ and thus $P_p^* > 0$ must hold, because $1-\varepsilon(1+\chi) > 1-\varepsilon\beta > 1-\alpha-\varepsilon\beta > 0$. When $1 > \varepsilon(1+\chi)$ holds, both $K_p^* > 0$ and $K_p^* < 0$ are possible. Under $K_p^* > 0$, the *kk* curve is increasing and the corresponding law of motion in a phase diagram indicates that an closed curve cannot exist around the unstable SE point. Next, under $K_p^* < 0$, $Det.J^* > 0$ implies that the *kk* and *pp* curves are both downward sloping with the *pp* curve steeper. Then the law of motion indicates that a closed curve can exist as shown in Fig.5-2.

certainly an equilibrium path. If the initial capital stock k_0 (which is not shown explicitly) lies above k_L and below k_H in Fig.5-2, the representative consumer chooses either an equilibrium path located within the closed orbit or the closed orbit itself. In the former case the equilibrium path converges to the SE point while in the latter case an economy gets into an endogenous periodic cycle.

Fig.5-2. Equilibrium Paths under *Sub*critical Hopf Bifurcation (Unstable periodic orbit)



Second, if a dynamic path starts from an initial point outside of the unstable closed orbit, it will go farther from this orbit with cyclical movements (see the dotted curve in Fig.5-2). First, it does not reach either the *k* axis or the *p* axis in a finite time. To see this, recall that p is equal to the marginal utility of consumption $(1/c)$, which is assumed to be positive. Therefore this dynamic path does not reach the *k* axis. In order to examine whether it will reach the *p* axis, we should ask whether the value of *k* will keep declining in the vicinity of the *p* axis (i.e., at $k \doteq 0$). Substituting $k \doteq 0$ into (15), we have $\dot{k}(t) \to \infty$, because $Ap^{\Omega}(1/k)^{-\Delta}$ goes to infinity under $1+\chi < \beta$ (which leads to $\Delta < 0$) for the Hopf bifurcation to occur. This means that when *k* decreases and enter the vicinity of the *p* axis, it will begin to increase. Thus this dynamic path will not reach the *p* axis.

Next, in order to examine whether this diverging path is an equilibrium, we need to separate two situations, depending on whether or not the TVC is satisfied along it. Here we consider the situation where at least one path starting from an initial point with k_0 below k_L and p_0 outside of the closed curve satisfies the TVC. Along this path diverging from the closed orbit, k keeps cyclical movements permanently, periodically approaching to zero. This sheds light on the possibility that an underdeveloped society where the initial capital stock is scarce will not be able to escape from this volatile economic fluctuations.

From a different viewpoint, we could add new understanding about the "corridor stability" by Benhabib and Miyao (1981) based on this analysis. The corridor stability means that the region of local stability in the neighborhood of the SE point is bounded by the closed orbit around it. Namely, when some exogenous shock changes an SE point, an economy will converge to a new SE point if the shock is small, but it will diverge from the new SE point if the shock is so large that the initial SE point lies outside of the closed orbit around the new SE point. This property will hold when the coordination of expectations is impossible and thus the representative consumer can switch to a different path on the way to the SE. However, when the coordination of expectations is possible (i.e., costs for the coordination and/or for a switching to a different path are not prohibitively high), we can find another possibility.

Suppose that capital k exceeds the critical level k_L on the diverging equilibrium path (the dotted curve in Fig. 5-2.) starting from k_0 below k_L. Then, an economy will be able to jump from this path to a point on the invariant closed curve or to a point on an equilibrium path (within the closed curve) converging to the SE point.[24] Then, even if an economy starts from a point outside of the closed curve, it can converge to an SE point. This means that even if the subcritical Hoph bifurcation happens, an economy can avoid the corridor stability when the coordination of expectations is possible. In other words, the "corridor stability" is likely to hold only when the costs for the coordination of expectation and/or for switching to a different path on the way are prohibitively high.

5.3 Nonexistence of Competitive Equilibrium Paths and New Type of "Underdevelopment Trap"

This subsection considers the situation where the TVC is not satisfied along all the dynamic paths starting from an initial point outside of the unstable closed curve.[25] Then no competitive equilibrium paths exist when an initial capital stock k_0 lies below k_L. I cannot find any rationales for eliminating this situation from the analysis, though it must be an extreme case.

What could be an economic interpretation of the nonexistence of equilibrium paths? In Fig.5-2, because the initial capital stock k_0 lies below k_L, you cannot find a competitive equilibrium path for a decentralized market economy. We could interpret

---

[24] I appreciate Professor Noritsugu Nakanish (Kobe University) and Professor Ken-ichi Akao (Waseda University) for discussions concerning this point. However, all remaining errors are mine.

[25] Imagine the situation where all such paths keep expanding in the northeastern direction from the closed orbit so fast that a product of $k(t)p(t)$ grows at a rate higher than $\rho > 0$.

this fact as implying that this underdeveloped society cannot establish a system of decentralized (competitive) market economy even if it intends to do it.

From a viewpoint of development policy, the interpretation above finds a new type of “underdevelopment trap”: an initial (or historical) shortage of capital accumulation may cause the failure to establish a system of market economy in underdeveloped countries. In such societies, the ”Big Push”, an exogenous increase in the initial capital stock k_0 up to at least k_L (but below k_H), will be able to shift from this regime to a regime where it can create and establish a decentralized market economy system. This insight on the role of the “Big Push” must also be a new finding of this paper.

Proposition 5: (Equilibrium Dynamics under Subcritical Hopf Bifurcation)

When a closed orbit emerges in the range of large values of the PC parameter ε ($\varepsilon_0 < \varepsilon$), the SE point is stable while the closed curve is orbitally unstable.

(i)An equilibrium path starting from the initial capital stock k_0 above k_L (and below k_H) will converge to the SE point or exhibit an endogenous periodic cycle.

(ii) If at least one diverging path starting from the initial capital stock k_0 below k_L satisfy the TVC, an economy keeps cyclical movements permanently with k periodically approaching to zero.

(iii) If all the diverging paths starting from the initial capital stock k_0 below k_L do not satisfy the TVC, no competitive equilibrium paths exist. That is, an underdeveloped society cannot establish a decentralized (competitive) market economy system even if it intends to do it. In this situation, the ”Big Push” by

increasing the initial capital stock k_0 *up to* k_L *(but below* k_H *) will help the economy shift to a regime where it can establish a decentralized market economy system.*

6. Dynamic Analysis of "Human Development Aid"

This section explores what kind of equilibrium path will be induced if this country accepts "human development aid (HDA)" from foreign countries. The reason for taking up HDA here is that there have been a lot of disputes and seriously opposing views among economists about whether foreign aid will play useful or harmful roles to promote the poverty reduction in the Millennium Development Goals (MDGs) since the beginning of the 21st century. However, those discussions do not seem to be based on the rigorous theoretical foundation on how an underdeveloped economy will evolve over time when foreign aid succeeds in poverty reduction. Rather, economists might implicitly have assumed that an underdeveloped economy will follow a unique equilibrium paths converging monotonically to an SE, as in the standard neoclassical growth theories.

An interesting question here must thus be whether we can always expect that an introduction of HDA will succeed in inducing an equilibrium path converging monotonically to an SE with higher welfare. If we can, we will be able to evaluate easily the success or failure of HDA. Otherwise, we should be more careful when evaluating it and suppose that a growth process induced by HDA may attain higher SE welfare even if it exhibits complicated (non-monotonic) movements.

To introduce the effect of HDA, we replace $h(c)$ with $\theta h(c) = \theta c^{\varepsilon}$ and assume an exogenous increase in parameter $\theta > 0$. It could be interpreted as this country receiving foreign aid because it incurs no cost for an increase in θ. In this formulation, each worker's labor productivity improves when the level of consumption remains unchanged. Examples of this kind of aid include cases when medical doctors from foreign countries teach people in developing countries an effective cure for dehydration or diseases, when specialists from advanced countries provide them with technological knowledge for obtaining clean water, or when an international development institution provides poor people suffering from malaria with mosquito nets for free.

In this modified model, we can rewrite (14) into：

$$n = \left[\theta^{\beta}(1-a)k^{\alpha} p^{1-\varepsilon\beta}\right]^{\frac{1}{1+\chi-\beta}} \tag{14'}$$

The equilibrium dynamic system is:

$$\dot{k}(t) = \theta^{\frac{1+\chi}{1+\chi-\beta}} Ak(t)^{\frac{\alpha(1+\chi)}{1+\chi-\beta}} p(t)^{\frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}} - \left(\frac{1}{p(t)}\right) - \delta k(t) = \tilde{K}(k, p : \theta) \tag{15'}$$

$$\dot{p}(t) = p(t)\left[\rho + \delta - \theta^{\frac{1+\chi}{1+\chi-\beta}} aAk(t)^{\frac{\alpha(1+\chi)}{1+\chi-\beta}-1} p(t)^{\frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}}\right] = \tilde{P}(k, p : \theta) \tag{16'}$$

An SE is defined by $\tilde{K}(k, p : \theta) = 0$ and $\tilde{P}(k, p : \theta) = 0$. Totally differentiating them yields:

$$\begin{pmatrix} \tilde{K}_k^{\,*} & \tilde{K}_p^{\,*} \\ \tilde{P}_k^{*} & \tilde{P}_p^{\,*} \end{pmatrix} \begin{pmatrix} dk \\ dp \end{pmatrix} = \begin{pmatrix} -\tilde{K}_\theta^{\,*} \\ -P_\theta^{\,*} \end{pmatrix} d\theta$$

The coefficient matrix evaluated at the SE is the same as J^* in section 2 ($K_k{}^* = \tilde{K}_k{}^*$, $K_p{}^* = \tilde{K}_p{}^*$, $P_k^* = \tilde{P}_k^*$, $P_p{}^* = \tilde{P}_p{}^*$). Furthermore, the partial derivatives with respect to θ will be:

$$\tilde{K}_\theta = \left(\frac{1+\chi}{1+\chi-\beta}\right)\theta^{\frac{1+\chi}{1+\chi-\beta}-1} Ak^{\frac{\alpha(1+\chi)}{1+\chi-\beta}} p^{\frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}}$$

$$\tilde{P}_\theta = p\left[-\left(\frac{1+\chi}{1+\chi-\beta}\right)\theta^{\frac{1+\chi}{1+\chi-\beta}-1} aAk^{\frac{\alpha(1+\chi)}{1+\chi-\beta}-1} p^{\frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}}\right]$$

The comparative-static results are：

$$\frac{dk^*}{d\theta} = \frac{\tilde{K}_p\tilde{P}_\theta - \tilde{P}_p\tilde{K}_\theta}{Det.J^*} \qquad \frac{dp^*}{d\theta} = \frac{\tilde{K}_\theta\tilde{P}_k - \tilde{P}_\theta\tilde{K}_k}{Det.J^*}$$

Let us look for situations where the welfare level in the SE

$$U^* = \frac{1}{\rho}\left[\log\frac{1}{p^*} - \frac{(n^*)^{1+\chi}}{1+\chi}\right]$$

improves by an introduction of HDA. The SE welfare U^* improves when p^* and n^* both decrease. To find that situation, we first see that

$$p^*k^* = \frac{a}{\rho + (1-a)\delta}$$

holds in an SE using (15') and (16'). Thus k^* and p^* move in the opposite directions. Next we derive a rate of change (represented by a hat notation) in n from (14'):

$$\hat{n}^* = \frac{1}{1+\chi-\beta}\left[\beta\hat{\theta} + \alpha\hat{k}^* + (1-\varepsilon\beta)\,\hat{p}^*\right] \tag{14''}$$

Under $1+\chi < \beta$, p^* and n^* both decrease if $\hat{k}^* > 0$ and $1-\varepsilon\beta < 0$ holds.

After investigating all the cases, we can find two cases when the SE welfare U^* unambiguously improves. The first case corresponds to a saddle-point stable SE shown in Fig.6-1 and the second to a perfectly stable SE shown in Fig.6-2. They are both based on $1+\chi<\beta$ and $1>\varepsilon(1+\chi)$, under which the *kk* and *pp* curves are decreasing. Then an increase in θ induces a downward shift both of the *kk* and of the *pp* curves. A new SE is located southeast of the initial SE (while k^* increase, p^* and n^* decrease. Thus the SE welfare is higher).

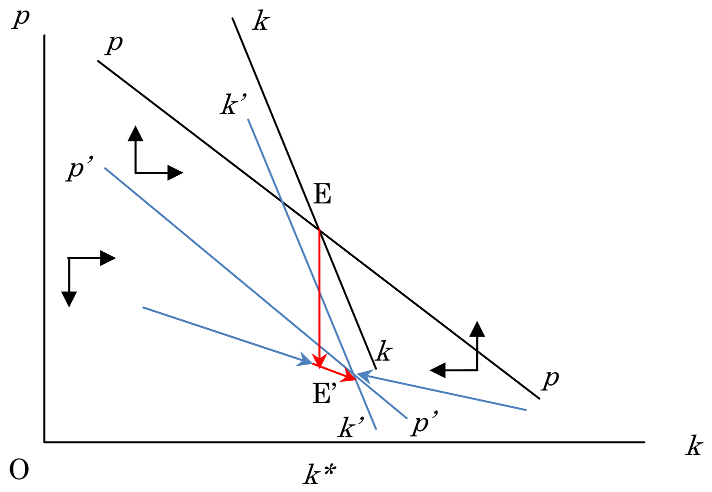
Let us confirm that they can actually happen. Solving $1+\chi<\beta$, $1>\varepsilon(1+\chi)$ and $1-\varepsilon\beta<0$ simultaneously for ε yields $(1/\beta)<\varepsilon<1/(1+\chi)$. When the value of ε lies in this range, the three inequalities hold simultaneously.

Before elucidating properties of equilibrium dynamics, we should pay attention to the fact that labor supply n^* decreases by an introduction of HDA. Even when the SE welfare improves, people facing the HDA intend to decrease their labor supply. It means that HDA will weaken an incentive for work effort while it induces capital accumulation and an increase in consumption.

6.1 Welfare Improving Aid for Saddle-point Stable SE

The first situation of the SE welfare improving HDA happens when the SE is saddle-point stable, shown in Fig. 6-1 (corresponding to Fig.3-2 in Appendix). When receiving HDA from abroad, an economy will jump on to the new transition path by increasing consumption temporarily. Then *k* begins to accumulate. During this process *c* increases along the path toward the new SE (E’). At point E’, it enjoys higher welfare than in the initial SE. This seems a situation for welfare improving aid that we naturally expect.

Fig.6-1. Welfare Improving HDA for Saddle-point stable SE ($1+\chi<\beta$, $1>\varepsilon(1+\chi)$ and $K_p^*<0$)



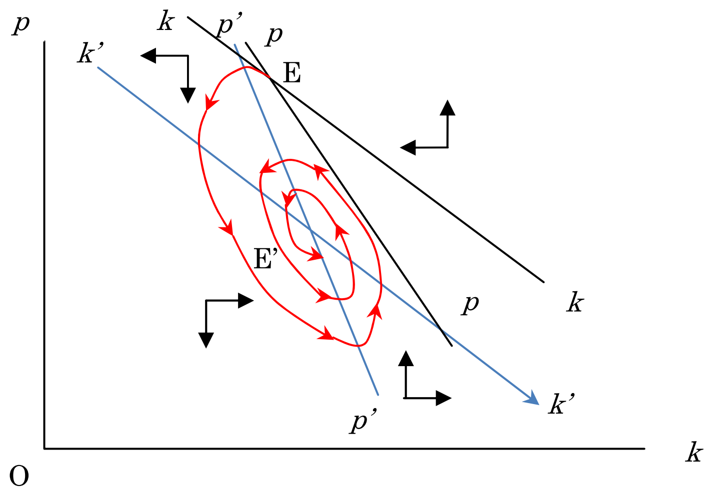
Furthermore, we should notice that welfare improves not only in the SE but also on the transition path in Fig. 6-1. To see it, recall $1+\chi<\beta$ and $1-\varepsilon\beta<0$ hold in (14'). A jump from point E means that p_t declines with k^* remaining constant. Thus labor supply n_t decreases. The instantaneous utility $u=\log(1/p_t)-n_t^{1+\lambda}/(1+\chi)$ rises at that point in time. On the transition path toward E', the instantaneous utility increases because k_t rises and p_t declines (n_t rises). Therefore an introduction of HDA improves total welfare on the whole process from the initial SE to the new SE.

6.2 Welfare Improving Aid for Perfectly Stable SE

The second situation of the SE welfare improving HDA happens when the SE is perfectly stable, shown in Fig. 6-2 (corresponding to Fig.3-4). When receiving HDA from abroad, *c* and *k* will begin to move toward the new SE by following a cyclical transition path. In this situation an economy can enjoy higher welfare in the SE.

However, it will exhibit complicated dynamics for consumption and capital stock. That is, at first both c and k begin to decrease, and then only c begins to rise. After that, k also increases during the consumption-increasing process, and then c begins to decrease while k increases. Further, both c and k decrease. Following this cyclical movement, an economy will reach the new SE, where it enjoys higher welfare than in the initial SE. (It is difficult to evaluate welfare along the transition path in Fig.6-2.)

Fig.6-2. Welfare Improving HDA for Perfectly Stable SE ($1+\chi<\beta$, $1>\varepsilon(1+\chi)$ and $K_p^*<0$)



Based on this result, we need to be careful when evaluating the effects of HDA. A poor developing economy may begin to exhibit complicated dynamics when HDA is introduced. It may not show simple monotonic movements from which we can regard the aid as successful and effective for an increase in consumption and capital accumulation. The economy which will attain higher welfare ultimately (in the new SE) may exhibit a decrease in consumption or capital depreciation before it reaches the new SE. In such a situation, we are inclined to interpret the HDA as a failure. However, the welfare in the new SE is certainly higher than before the aid comes in (if time discount rate ρ is very low, the total welfare may increase).

Finally, let us explain three effects of an increase in θ. The first and the second effects are at work even if the PC externality is absent while the third is specific to the PCH. First, an increase in θ, other things being equal, raises labor input in efficiency units $\theta h(c)n$ and thus the output. The saving and investment increase, given the initial level of consumption. Because of the corresponding capital accumulation, the rental rate r of capital declines along the marginal product of capital (MPK) curve. By Euler equation, this leads to $\dot{p} > 0$, and thus p^* tends to rise while consumption c^* to decrease. The second effect comes from the upward shift of the MPK curve due to an increase in $\theta h(c)n$. This upward shift leads to a rise in the rental rate $r = ak^{\alpha-1}[\theta h(c)n]^{\beta}$. Thus p^* tends to decline while consumption c^* to increase. These two effects, as you see, work in the opposite directions.

The third effect specific to the PCH will affect the labor market equilibrium through a change in the PC function $h(c) = c^{\varepsilon}$. Note that given c, the labor demand curve $w = (1-a)k^{\alpha}[\theta h(c)n]^{\beta-1}$ is steeper than the increasing labor supply curve $w = c^{1-\varepsilon}n^{\chi}$ in Fig.6-1 and 6-2 (because of $\chi < \beta - 1$). Consider the situation where the first effect mentioned above dominates and thus consumption c decreases. Then a decrease in c makes the labor demand curve shift down because $h(c)$ decreases. Given an increasing labor supply curve $w = c^{1-\varepsilon}n^{\chi}$, labor input n tends to expand. However, in Fig.6-1 and 6-2 where $1 > \varepsilon(1+\chi)$ holds and thus the concavity of $h(c) = c^{\varepsilon}$ is relatively strong, one unit decrease in consumption induces a smaller decline in $h(c)$.

Therefore efficiency labor $\theta h(c)n$ tends to increase. Since $r = ak^{\alpha-1}[\theta h(c)n]^{\beta}$ tends to rise, c^* in the SE tends to increase. (This tendency will be offset partially if we take account of the downward shift of the labor supply curve $w = c^{1-\varepsilon}n^{\chi}$.) In this sense, the PC externality contributes to an increase in consumption and welfare improvement by an introduction of HDA.

Proposition 6 (Characterization of Equilibrium Dynamics due to Welfare Improving Human Development Aid): *An introduction of human development aid (HDA)can improve welfare in the SE under* $1+\chi < \beta$ *and* $1 > \varepsilon(1+\chi)$.

(i)In the case of saddle-point stable SE in Fig.6-1, an economy will jump on to the new transition path by temporarily increasing consumption. Then capital stock k accumulates and consumption c increases along the equilibrium path toward the new SE (E'). Total welfare both in the SE and along the transition path improves.

(ii) In the case of perfectly stable SE in Fig.6-2, consumption and capital stock will begin to move toward the new SE by following a cyclical transition path. An economy enjoys higher welfare in the new SE. However, the welfare effect along the transition path is ambiguous.

(iii) Even if an introduction of HDA improves the SE welfare, it weakens an incentive for labor supply in the recipient country, decreasing their working hours.

7. Conclusions

This paper has investigated macroeconomic dynamics of underdeveloped societies in which an increase in consumption improves worker's productivity. To construct a tractable and general model, we introduce the PCH (Steger (2002)) into a standard one-sector RBC model with factor-generated externalities (Benhabib and farmer (1994)). First, indeterminacy happens due to the PC externality even in the absence of capital externality. Second, in contrast to the standard growth models, the "intertemporal *complementarity* between present and future consumption" may hold under a saddle-point stable SE. Third, under indeterminacy, the PC effect contributes to generate cyclical behaviors of equilibrium paths converging to an SE. When the *super*critical Hoph bifurcation occurs, equilibrium paths may either diverge from an SE point or exhibit an endogenous cycle. These non-monotonic behaviors are consistent with empirical data. Forth, when the *sub*critical Hoph bifurcation occurs, even when initial capital falls short of a critical level, we can avoid the "corridor stability" if coordination of expectation is possible. If it is impossible, an economy keeps cyclical divergence permanently with capital periodically approaching to zero. When no competitive equilibrium paths exist, it could be interpreted as a failure to establish a system of decentralized (competitive) market economy. The "Big Push" may help escape from this new type of "underdevelopment trap". Finally, an introduction of human development aid can induce a cyclical path converging to a new SE with higher welfare. We cannot always expect an underdeveloped society to follow a monotonic growth path even if the aid improves SE welfare.

Finally let us point out future agenda. First, an extension to an open economy model must be important, because international trade and capital movements play an important role even in underdeveloped countries. Second, we could proceed to a two or three sector model, in which we will be able to derive more realistic conditions for indeterminacy. Third, it may be interesting to explore consequences of an introduction of a population growth rate.

Appendix A. Equilibrium Dynamics for Saddle-point Stable SE

Section 3 focuses on the case of $1 > \varepsilon > 1/(1+\chi)$ under $1+\chi > \beta$. In this Appendix, we show the case of $1 > \varepsilon(1+\chi)$ under $1+\chi > \beta$ and the cases of $1+\chi < \beta$.[26]

A.1 Cases of $1 > \varepsilon(1+\chi)$ under $1+\chi > \beta$:

Under $1+\chi > \beta$, we have qualitatively the same equilibrium paths for $1 > \varepsilon(1+\chi)$ as those in Fig.2-1 in the text. In both cases shown in Fig. 1-1 and 1-2, $K_p^* > 0$ and thus $-(K_k^* / K_p^*) < 0$ holds (the *kk* curve is decreasing). By $Det.J^* < 0$, we get $-(K_k^* / K_p^*) <$ $-(P_k^* / P_p^*)$ (the slope of the *pp* curve is larger than that of the decreasing *kk* curve). Since $P_p^* < 0$ holds but the sign of P_k^* is ambiguous, the *pp* curve can be either increasing or decreasing. Fig. 1-1 shows the case of $P_k^* > 0$ while Fig. 1-2 that of $P_k^* < 0$. In either case, an equilibrium path will be uniquely determined if it starts from the low level k_0 of initial capital stock. Per capita consumption c and capital k both increase monotonically along the path, converging to the SE.

---

[26] Under $1+\chi < \beta$, an SE can be saddle-point stable both for $1 > \varepsilon(1+\lambda)$ and for $1 < \varepsilon(1+\lambda)$.

Fig.1-1.Saddle-point Stable SE under $1+\chi > \beta$, $1 > \varepsilon(1+\chi)$ and $P_k^* > 0$

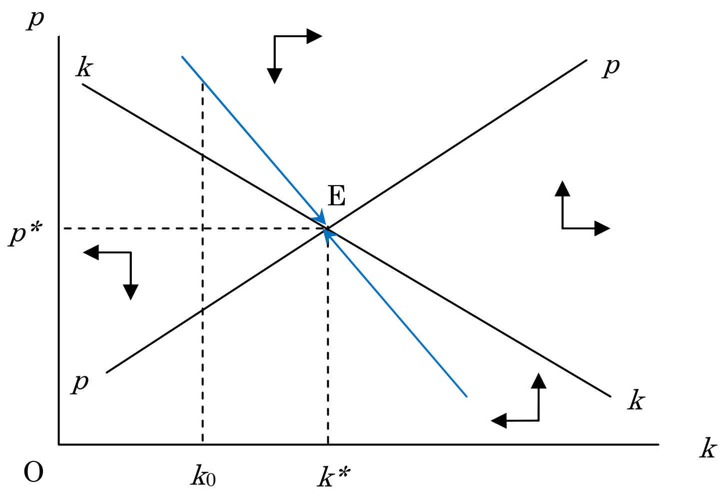
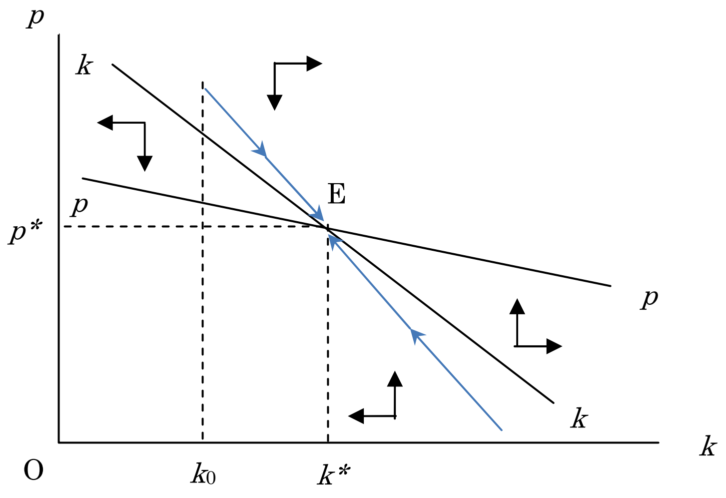


Fig. 1-2. Saddle-point Stable SE under $1+\chi > \beta$, $1 > \varepsilon(1+\chi)$ and $P_k^* < 0$



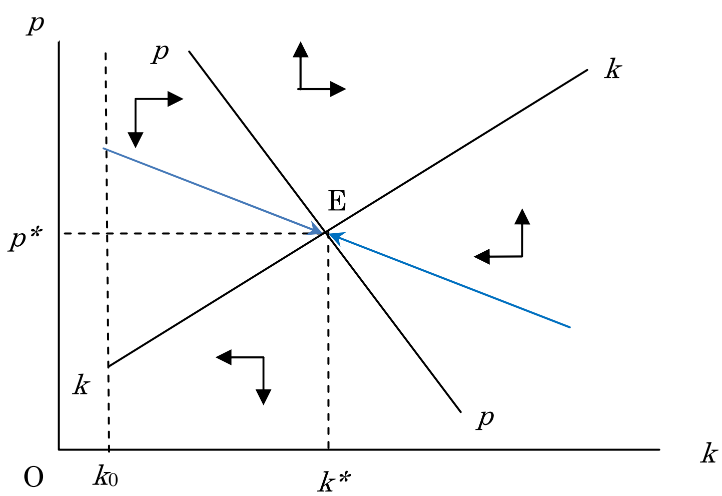
A.2 Cases under $1+\chi < \beta$ *:*

Under $1+\chi < \beta$, $Det.J^* = K_k^* P_p^* - K_p^* P_k^* < 0$ is equivalent to $1-\alpha-\varepsilon\beta < 0$, i.e., $(1-\alpha)/\beta < \varepsilon$ (see (22)). That is, an SE will be saddle-point stable when the capital externality α is large and/or ε is large (the PC function $h(c)$ is weakly concave, i.e., close to linear, or convex). We will separate three different cases, but the property of equilibrium paths is qualitatively the same among them. For the analysis below, note that we always have $K_k^* < 0$ and $P_k^* > 0$ under $1+\chi < \beta$.

A.2-1. Cases of $1 > \varepsilon(1+\chi)$ *under* $1+\chi < \beta$ *:*

When $1 > \varepsilon(1+\chi)$ holds under $1+\chi < \beta$, $P_p^* > 0$ always holds. This is consistent with $Det.J^* < 0$ because $(1-\alpha)/\beta < \varepsilon < 1/(1+\chi)$ holds. The sign of K_p^* is ambiguous.[27] However, so long as the SE is saddle-point stable, the property of transition equilibrium paths is the same, regardless of whether $K_p^* > 0$ or $K_p^* < 0$ holds (See Appendix B for confirming these cases are possible).

Fig.3-1. Saddle-point Stable SE under $1+\chi < \beta$, $1 > \varepsilon(1+\chi)$ and $K_p^* > 0$



[27] These sign conditions are the same as in the case of $\varepsilon = 0$. Thus the qualitative results for $1 > \varepsilon(1+\chi)$ will remain in the model without the PCH.

Fig.3-1 shows the case when $K_p^* > 0$ (and $Det.J^* < 0$ always) holds. Then, we get $-(K_k^* / K_p^*) > 0$ (the *kk* curve is increasing) and $-(P_k^* / P_p^*) < 0$ (the *pp* curve is decreasing). Thus an SE exists uniquely, and an equilibrium path is uniquely determined.

Fig.3-2. Saddle-point Stable SE under $1+\chi < \beta$, $1 > \varepsilon(1+\chi)$ and $K_p^* < 0$

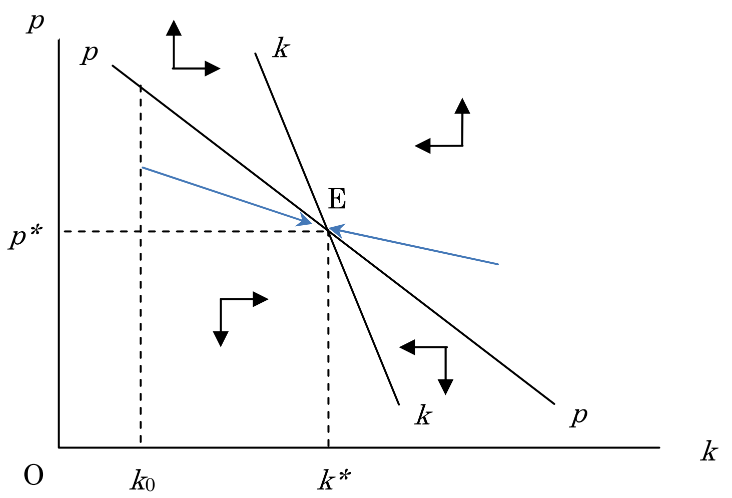
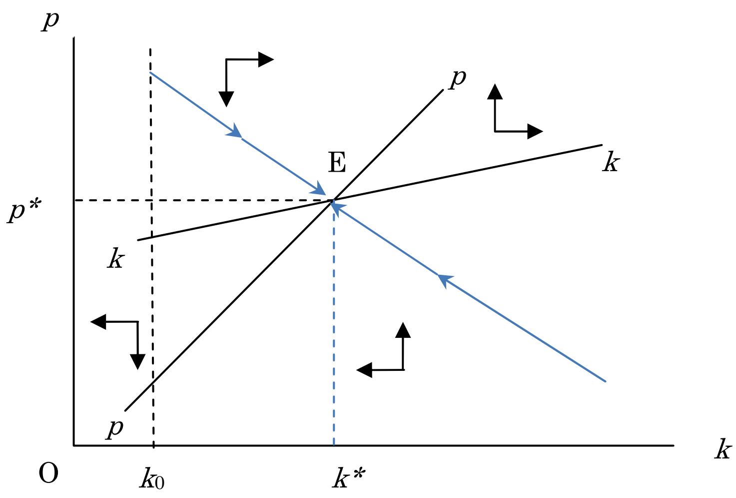


Fig.3-2 shows the case of $K_p^* < 0$. Because $Det.J^* = K_k^* P_p^* - K_p^* P_k^* < 0$ is equivalent to $-(K_k^* / K_p^*) < -(P_k^* / P_p^*) < 0$, the *kk* and the *pp* curves are both decreasing with the *kk* curve steeper. Therefore an equilibrium path for $Det.J^* < 0$ is shown in Fig.3-2.

A.2-2. Case of $\varepsilon(1+\chi) > 1$ *under* $1+\chi < \beta$

Fig.3-3 shows the case of $\varepsilon(1+\chi) > 1$ under $1+\chi < \beta$. Then $1-\alpha-\varepsilon\beta < 1-\alpha -\varepsilon(1+\chi) = [1-\varepsilon(1+\chi)] - \alpha < 0$ holds and thus, by (22), $Det.J^* < 0$ holds. Taking account of $K_p^* > 0$ and $P_p^* < 0$, the *kk* and *pp* curves are both increasing with the *pp* curve steeper, because $Det.J^* < 0$ implies $0 < -(K_k^*/K_p^*) < -(P_k^*/P_p^*)$. An equilibrium path is shown in Fig.3-3.

Fig.3-3. Saddle-point Stable SE under $1+\chi < \beta$ and $\varepsilon(1+\chi) > 1$



Let us explain intuitively why the equilibrium dynamics above occurs under $1+\chi < \beta$. Recall that the labor demand curve is steeper than the increasing labor supply curve ($\chi < \beta - 1$). When the initial consumption $c_0 = 1/p_0$ is low, the labor demand curve lie in a low position (because of a large value of β, given ε) and thus the equilibrium labor input n tends to be large. Thus efficiency labor input $h(c)n$ tends to be large, too. Because this leads to a high value of r, Euler equation (12) indicates that p declines and therefore c increases along the transition path.

Appendix B: Compatibility of Conditions for Saddle-point Stable Cases

B-1. Possibility of Fig.2-1 and Fig.2-2 in the text

We will examine whether $K_p^* > 0$ ($K_p^* < 0$) actually holds under $1+\chi > \beta$ and $\varepsilon(1+\chi) > 1$. Condition $K_p^* > 0$ ($K_p^* < 0$) is equivalent to

$$\varepsilon(1+\chi) < (>) 1 + \left(\frac{1+\chi-\beta}{\beta}\right)\left[\frac{\rho+(1-a)\delta}{\rho+\delta}\right]$$

Because the right-hand side is larger than unity, this inequality can be compatible with $\varepsilon(1+\chi) > 1$ and thus $K_p^* > 0$ is possible. The inequality directly implies $\varepsilon(1+\chi) > 1$, and thus $K_p^* < 0$ is consistent with $\varepsilon(1+\chi) > 1$.

B.2 Possibility of Fig.3-1 and Fig.3-2 : in Appendix

We will examine whether $K_p^* > 0$ ($K_p^* < 0$) actually holds under $1+\chi < \beta$ and $1 > \varepsilon(1+\chi)$. Condition $K_p^* > 0$ ($K_p^* < 0$) is equivalent to

$$\varepsilon(1+\chi) < (>) 1 + \left(\frac{1+\chi-\beta}{\beta}\right)\left[\frac{\rho+(1-a)\delta}{\rho+\delta}\right]$$

Because the right-hand side is smaller than unity, this inequality directly implies $1 > \varepsilon(1+\chi)$, and thus $K_p^* > 0$ is consistent with $1 > \varepsilon(1+\chi)$. $K_p^* < 0$ can be compatible with $1 > \varepsilon(1+\chi)$.

Appendix C: Proof of the Hopf Bifurcation

C.1 The Hoph Bifurcation Theorem:

Guckenheimer and Holmes (1983, pp.151-152) provide a sufficient condition for the Hopf bifurcation in Theorem 3.4.2. Here we use the trace and determinant conditions provides in proposition 4.1 in Yoshida (2003, p.78). Let us summarize the theorem for a 2-dimensional dynamical system.

Hopf Bifurcation Theorem (Yoshida (2003)):

Suppose that the system $\dot{x} = f(x;\varepsilon)$, $x \in \mathbb{R}^n$ *with a bifurcation parameter* $\varepsilon \in \mathbb{R}$ *has an equilibrium* (x^*, ε_0) *at which the following properties (H-1), (H-2) and (H-3) are satisfied.*

(H-1) Equilibrium value $x^*(\varepsilon)$ *of this system is a smooth function of* ε.

Denote the Jacobian matrix of $f(x;\varepsilon)$ *evaluated at the equilibrium by* $J^*(\varepsilon)$. *When* $n = 2$ *holds,*[28]

(H-2) $Trace.J^*(\varepsilon_0) = 0$ *and* $Det.J^*(\varepsilon_0) > 0$

(H-3) $d[Trace.J^*(\varepsilon_0)]/d\varepsilon \neq 0$

Then there exists a periodic solution which bifurcates from $x^*(\varepsilon_0)$ *at* $\varepsilon = \varepsilon_0$, *and its amplitude is given by* $2\pi / \operatorname{Im}\lambda(\varepsilon_0)$ *approximately.* □

C.2 Proof of Existence of Periodic Solution:

Focusing on the model with no capital externality ($\alpha = a$ and $\beta > 1 - a$), we show that a

[28] The characteristic equation for $n = 2$ is $\lambda^2 - (Trace.J^*(\varepsilon))\lambda + Det.J^*(\varepsilon) = 0$.

change in the PC parameter ε induces the Hoph bifurcation.

(H-1): An SE is defined by $K(k^*, p^*; \varepsilon) = 0$ and $P(k^*, p^*; \varepsilon) = 0$. Since $Det.J^* =$ $K_k^* P_p^* - K_p^* P_k^* \neq 0$ holds, by the implicit function theorem, there exist smooth functions $k^*(\varepsilon)$ and $p^*(\varepsilon)$ that satisfy $K(k^*(\varepsilon), p^*(\varepsilon); \varepsilon) = 0$ and $P(k^*(\varepsilon), p^*(\varepsilon); \varepsilon) = 0$. Therefore the SE values $(k^*(\varepsilon), p^*(\varepsilon))$ are smooth functions of ε.

(H-2): Under $1+\chi < \beta$, the value of ε that satisfies

$$Trace.J^* = \rho + \frac{\varepsilon\beta(1+\chi)(\rho+\delta)}{1+\chi-\beta} = 0$$

is

$$\varepsilon_0 = \left(\frac{\rho}{\rho+\delta}\right)\left[\frac{1}{1+\chi} - \frac{1}{\beta}\right] > 0$$

As was already explained in the text, if the value of δ is so large that

$$\rho + \delta > \left(\frac{\rho}{1-a}\right)\left[\frac{\beta}{1+\chi} - 1\right] \tag{K}$$

holds, $Det.J^* > 0$, which is equivalent to $\varepsilon < \frac{1-a}{\beta}$, is satisfied at $\varepsilon = \varepsilon_0$.

(H-3): Under $1+\chi < \beta$, we get

$$\frac{d[Trace.J^*(\varepsilon_0)]}{d\varepsilon} = \frac{\beta(1+\chi)(\rho+\delta)}{1+\chi-\beta} < 0$$

This term is not zero. Therefore, a periodic solution (an invariant closed curve) exists around the SE point.

Appendix D: The Speed of Convergence

In the text we have derived *local* properties of equilibrium paths by linear approximation in the neighborhood of an SE point. Nevertheless, we could interpret them as having realistic relevance in the sense that an economy will spend a relatively long time moving along a path converging to the SE point under plausible values of exogenous parameters.

Let us quantitatively evaluate the speed of convergence to the SE point. Following the method by Romer (2006), we will reformulate the linearized system into

$$d(k_t - k^*)/dt = K_k^*\left[k_t - k^*\right] + K_p^*\left[p_t - p^*\right]$$

$$d(p_t - p^*)/dt = P_k^*\left[k_t - k^*\right] + P_p^*\left[p_t - p^*\right]$$

Here we use $\dot{k}_t = d(k_t - k^*)/dt$ and $\dot{p}_t = d(p_t - p^*)/dt$, because the SE values are constant. Focusing on the path along which $(k_t - k^*)$ and $(p_t - p^*)$ change at the same rate $\mu = \dfrac{d(k_t - k^*)/dt}{k_t - k^*} = \dfrac{d(p_t - p^*)/dt}{p_t - p^*}$ over time,we obtain $\dfrac{k_t - k^*}{p_t - p^*} = \dfrac{K_p^*}{\mu - K_k^*}$.

Substituting this into (A-) and rearranging the terms, we get

$$\mu^2 - (Trace.J^*)\mu + Det.J^* = 0.$$

This quadratic equation have two roots $\mu_1 < 0$ and $\mu_2 > 0$. We will examine the absolute value of

$$\mu_1 = \frac{1}{2}\left[Trace.J^* - \sqrt{(Trace.J^*)^2 - 4Det.J^*}\right]$$

which corresponds to the saddle-point stable SE ($Det.J^* < 0$).

We will evaluate the speed of convergence in the model with $\alpha = a$ under realistically relevant parameter values $\rho = 0.02$, $\delta = 0.03$. Recall the constraints for

β and ε, and then derive the value of $\mu_1 < 0$ for some numerical examples.

First, the parameters β and ε must satisfy three constraints for the saddle-point stable SE; (i) $\beta > 1-a$, (ii) $1+\chi > \beta$ and (iii) $1-a-\varepsilon\beta > 0$. Setting $\chi = 0$, we have $1-a < \beta < 1$. Thus $\varepsilon < 1$ is necessary for (iii). Next, we use the usual value of $a = 0.33$, and thus $0.67 < \beta < 1$ holds. If we set $\beta = 0.9$, we have $\varepsilon < (1-a)/\beta = 0.67/0.9 = 0.7\dot{4}$ from (iii). Let us show six numerical examples. Examples 1 corresponds to a high value of $\beta = 0.9$ while Example 2 to a high value of $\varepsilon = 0.98$ (the corresponding low value of $\beta = 0.68$). Examples 3 to 6 have the same intermediate value of $\beta = 0.8$. We obtain the speed of convergence μ_1 as follows.

Ex.1 : $\beta = 0.9$ and $\varepsilon = 0.74$ $\quad \mu_1 = -0.022\%$ $\quad \hat{t} = 3159.34$

Ex.2 : $\beta = 0.68$ and $\varepsilon = 0.98$ $\quad \mu_1 = -0.1092\%$ $\quad \hat{t} = 631.87$

Ex.3 : $\beta = 0.8$ and $\varepsilon = 0.6$ $\quad \mu_1 = -3.3199\%$, $\quad \hat{t} = 20.78$

Ex.4 : $\beta = 0.8$ and $\varepsilon = 0.65$ $\quad \mu_1 = -2.582\%$, $\quad \hat{t} = 26.72$

Ex.5 : $\beta = 0.8$ and $\varepsilon = 0.7$ $\quad \mu_1 = -1.849\%$, $\quad \hat{t} = 37.86$

Ex.6 : $\beta = 0.8$ and $\varepsilon = 0.8$ $\quad \mu_1 = -0.4916\%$, $\quad \hat{t} = 140.37$

The half-life $\hat{t}$ of the distance from a SE point to an initial point is calculated as $\hat{t} = -\log 2/(0.01)\mu_1 = -69/\mu_1$ (note that μ_1 is represented in percentage points). For example, the half-life for Example 3 implies that an economy will spend 20.78 years until it runs half a distance from the SE point to the initial point. These examples seem to justify our focus on transition paths toward an SE point.

***Acknowledgements:**

This paper was presented at the Australasian Meeting 2014 (Tasmania, Australia) and the Asian Meeting 2013 of the Econometric Society (Singapore), the 10th Biennial Pacific Rim Conference of Western Economic Association International (Keio University) and the 19th Decentralization Conference (Yokohama National University). I appreciate important discussions of Professor Kazuo Mino (Kyoto University), Takumi Naito (Waseda University), Masatoshi Tsumagari (Keio University), Noritugu Nakanishi and Takashi Kamihigashi (Kobe University). I thank Professor Michihiro Ohyama (Keio University, Emeritus), Tatsuro Iwaisako (Osaka University), Seiichi Katayama (Aichi Gakuin University), Hiroyuki Ozaki (Keio University), Yoshimasa Shirai (Keio University), Akira Momota (University of Tsukuba), Masamichi Kawano, Keisaku Higashida (Kansei Gakuin Unversity) for useful discussions. Comments to an earlier version of this paper from Professor Koichi Futagami (Osaka University), Takayuki Tsuruga (Kyoto University) are also useful and appreciated. All remaining errors are mine. I appreciate for the financial supports by Grant-in-Aid for Scientific Research (Basic Research Program (B) 21330070)) and Basic Research Program (C) 24530241 of Japan Society for the Promotion of Science (JSPS), and by Keio Gijuku Academic Development Funds.

References

Arrow, K.J., 1962. The economic implications of learning by doing. Review of Economic Studies 29, 155-173.

Banerji, S., Gupta, M.R., 1997. The efficiency wage given long-run employment and concave labor constraint. Journal of Development Economics 53, 185–195.

Becker,G.S., Murphy,K.M. and Tamura,R., 1990, Human capital, fertility and economic growth, Journal of Political Economy Vol.98, No.5, October, Part 2, pp.S12-S37.

Benhabib, J., Farmer, R.E.A., 1994, Indeterminacy and increasing returns. Journal of Economic Theory 63, 19–41.

Benhabib, J., Miyao, T., 1981, Some new results on the dynamics of the generalized Tobin model. International Economic review 22, 589–596.

Benhabib, J., Nishimura, K., 1998, Indeterminacy and sunspot with constant returns. Journal of Economic Theory 81, 58–96.

Bennett, R.L. and R.E. Farmer, 2000, Indeterminacy with Non-separable Utility. Journal of Economic Theory 93, 118–143.

Bliss, C., Stern, N., 1978. Productivity, wages and nutrition part I: theory. Journal of Development Economics 5, 331–362.

Daitoh, I., 2010. Productive consumption and population dynamics in an endogenous growth model: demographic trends and human development aid in developing economies. Journal of Economic Dynamics and Control 34, 696-709.

Dasgupta, P., Ray, D., 1986. Inequality as a determinant of malnutrition and underemployment: theory. Economic Journal 96, 1011–1034.

Easterly, W., 2006. The white man's burden: why the West's efforts to aid the rest have done so much ill and so little good. The Wylie Agency (UK), Ltd. (Japanese translation)

Gersovitz, M., 1983. Savings and nutrition at low income. Journal of Political Economy 91, 841–855.

Guckenheimer ,J. and P. Holmes, 1983. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields (Applied Mathematical Sciences). Springer-Verlag, New York.

Gupta, M.R., 2003. Productive consumption and endogenous growth: a theoretical analysis. Keio Economic Studies 40, 45–57.

Fukao, K., Benabou, R., 1993. History versus expectations: a comment. Quarterly Journal of Economics May, 535–542.

Jellala, M., Zenoub, Y., 2000. A dynamic efficiency wage model with learning by doing. Economics Letters 66, 99–105.

Krugman, P.A., 1991. History versus expectations. Quarterly Journal of Economics, May, 651–667.

Leibenstein, H., 1957a. Economic Backwardness and Economic Growth. Wiley, New York.

Leibenstein, H., 1957b. The theory of underemployment in backward economies. Journal of Political Economy 65, 91-103.

Lucas, R.E., 1993, "Making a Miracle," Econometrica Vol.61, No.2 (March), 251-272.

Matsuyama, K., 1991. Increasing returns, industrialization and indeterminacy of equilibrium. Quarterly Journal of Economics CX, 857-880.

McGough,B., Q. Meng, J.Xue, 2013. Expectational stability of sunspot equilibria in non-convex economies. Journal of Economic Dynamics and Control 37, 1126-1141.

Meng, Q, Yip, C.K., 2008. On indeterminacy in one-sector models of the business cycle with factor-generated externalities. Journal of macroeconomics 30, 97-110.

Mino, K, 1999. Non-separable utility function and indeterminacy of equilibrium in a model with human capital. Economics Letters 62, 311-317.

Mino, K, 2001. Indeterminacy of equilibrium and endogenous growth with social constant returns. Journal of Economic Theory 97, 203-222.

Ray, D., Streufert, P., 1993. Dynamic equilibria with unemployment due to undernourishment. Economic Theory 3, 61–85.

Sachs, J.D., 2005. The end of poverty: how we can make it happen in our lifetime. The Wylie Agency (UK), Ltd. (Japanese translation)

Steger, T.M., 2000. Productive consumption and growth in developing countries. Review of Development Economics 4, 365–375.

Steger, T.M., 2002. Productive consumption, the intertemporal consumption trade-off and growth. Journal of Economic Dynamics and Control 26, 1053–1068.

Stiglitz, J., 1976. The efficiency wage hypothesis, surplus of labor and distribution of income in the LDCs. Oxford Economic Papers 28, 185–207.

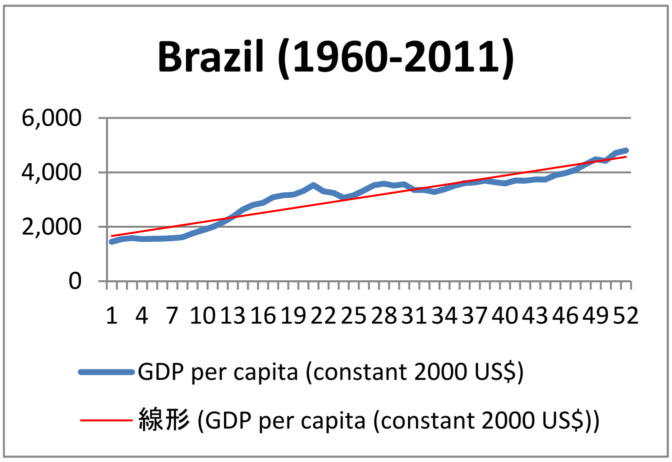
Wen,Y., 1998. Capacity utilization under increasing returns to scale. Journal of Economic Theory 81, 7-36.

Yoshida, H., 2003. Theory of Business Cycle: Non-linear Dynamic Approach, Nagoya University Press. (Japanese).

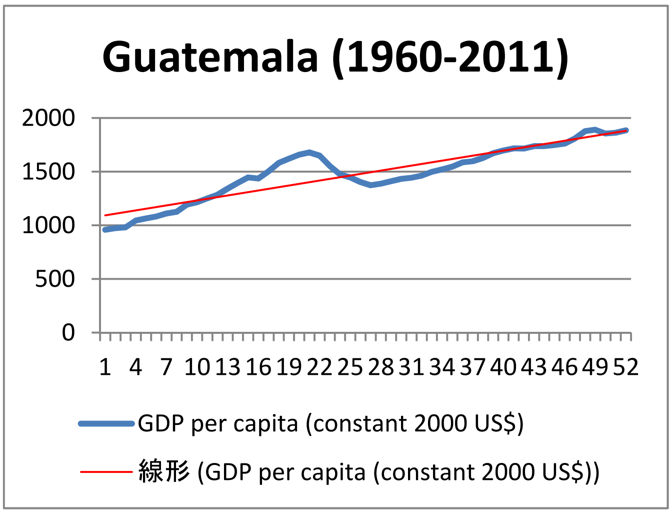
Table 1: The Millennium Development Goals

1. Eradicate extreme poverty and hunger
2. Achieve universal primary education
3. Promote gender equality and empower women
4. Reduce child mortality
5. Improve maternal health
6. Combat HIV/AIDS, malaria, and other diseases
7. Ensure environmental sustainability
8. Develop a global partnership for development

Tables: Indeterminacy Cases (Latin America)



Horizontal axis: the number of years (e.g., The 10th point on the horizontal axis is for 1969.)



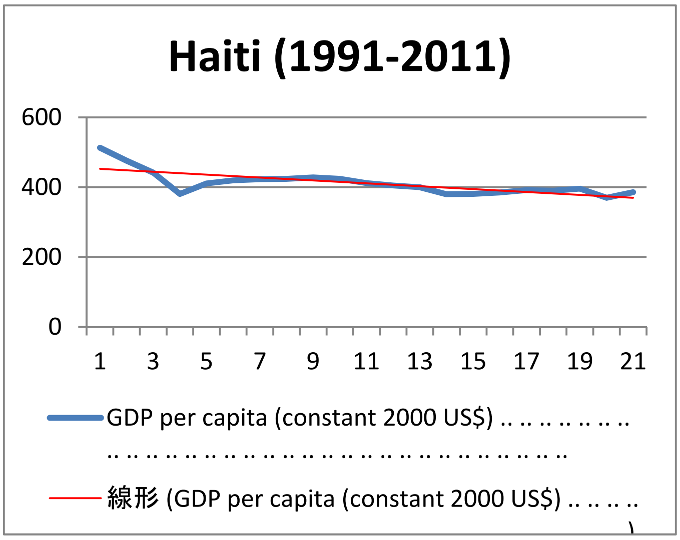
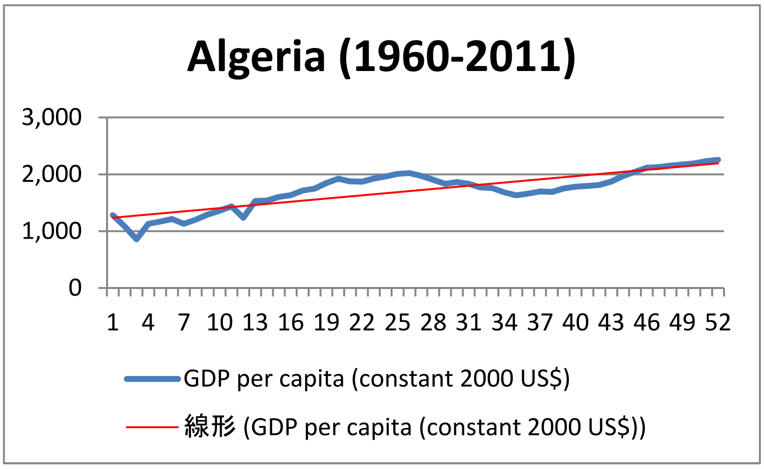
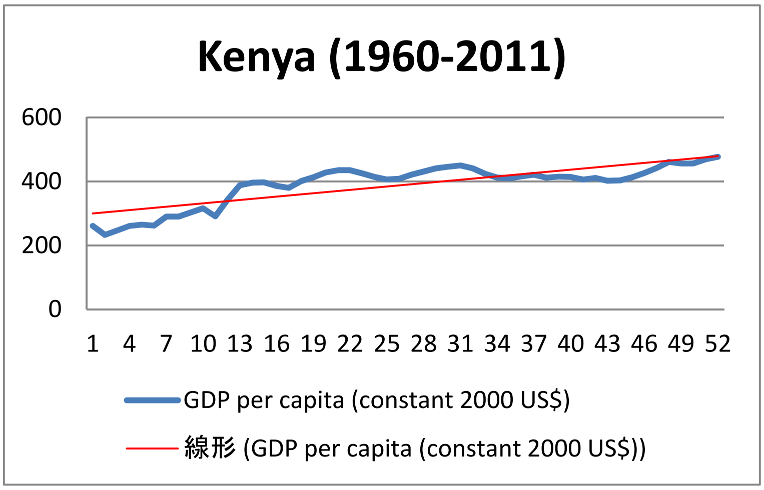
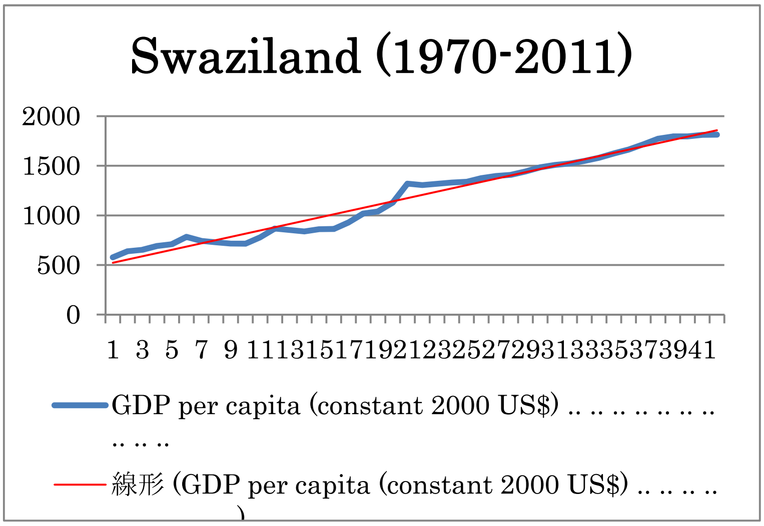


Table: Indeterminacy Cases (Africa)

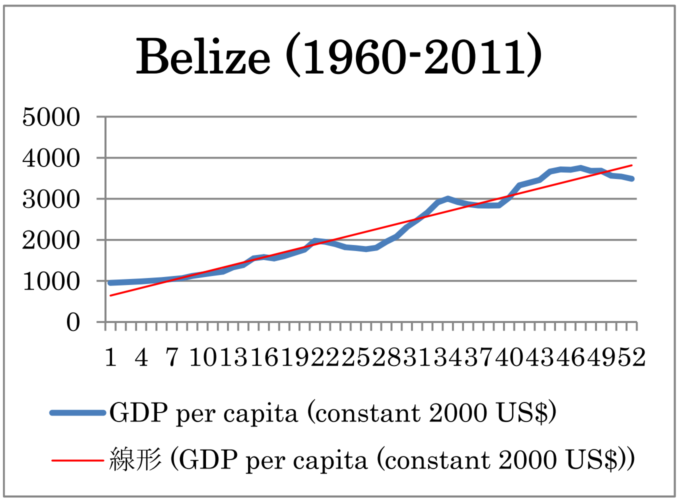




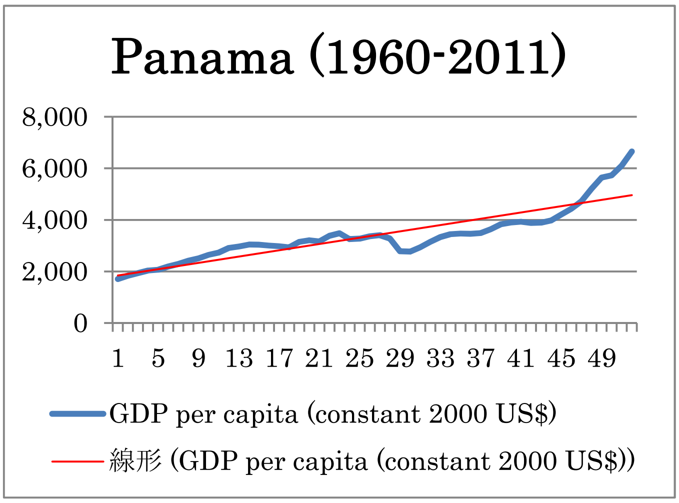


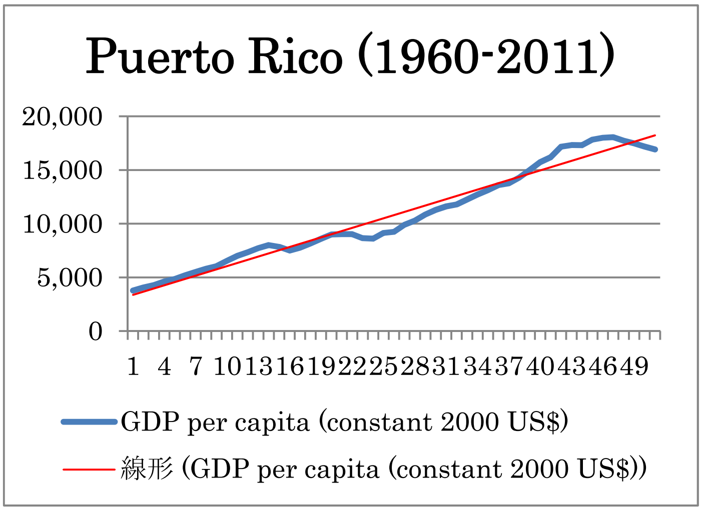
(Source: World Development Indicators)

Tables: Bifurcation Cases (Latin America)

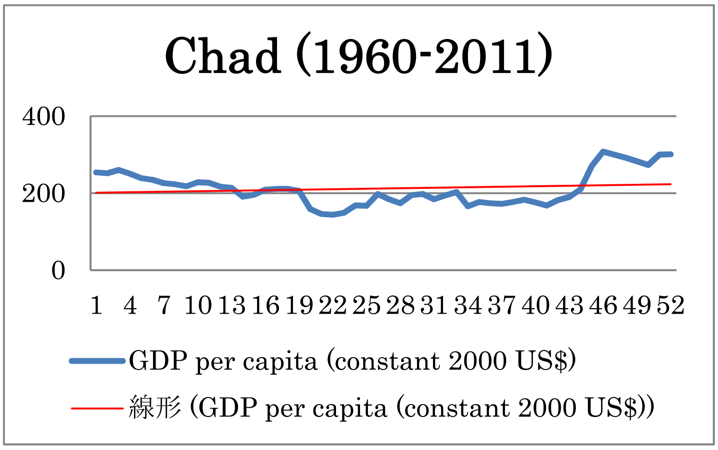


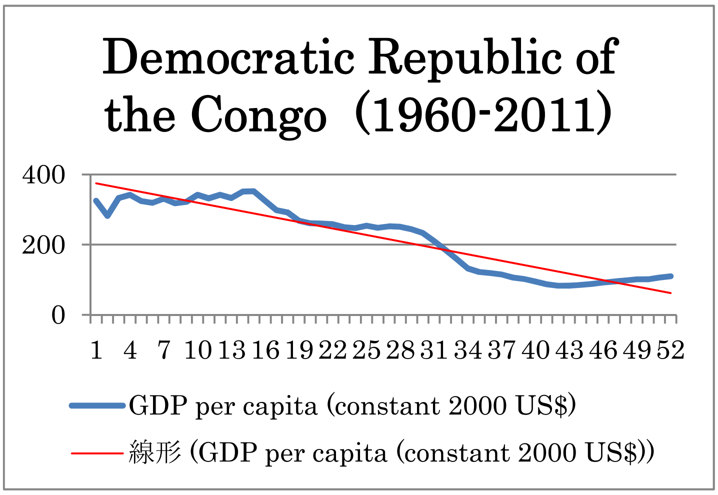
Horizontal axis: the number of years (e.g., The 10th point on the horizontal axis is for 1969.)

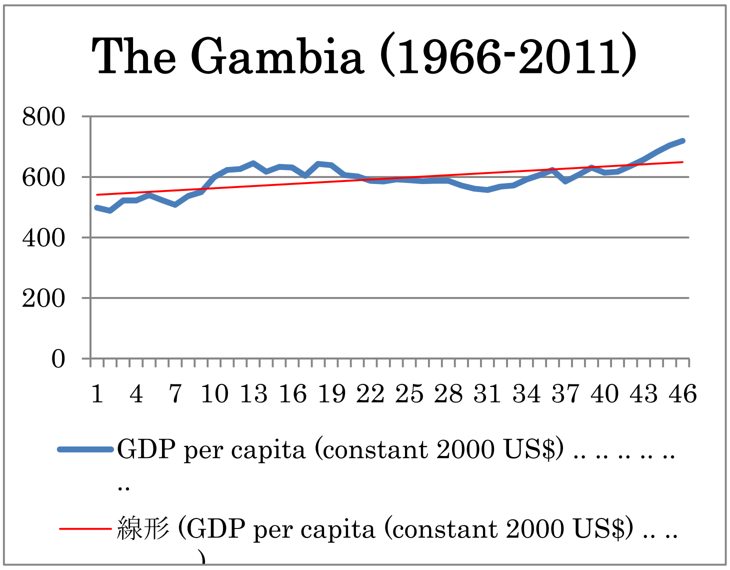


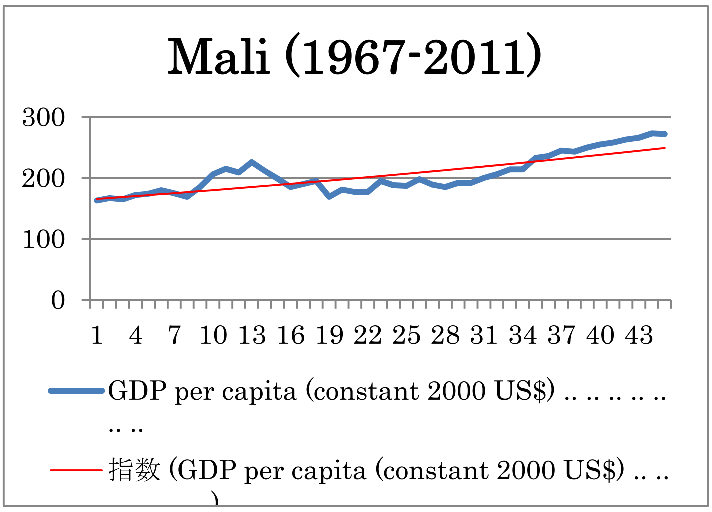


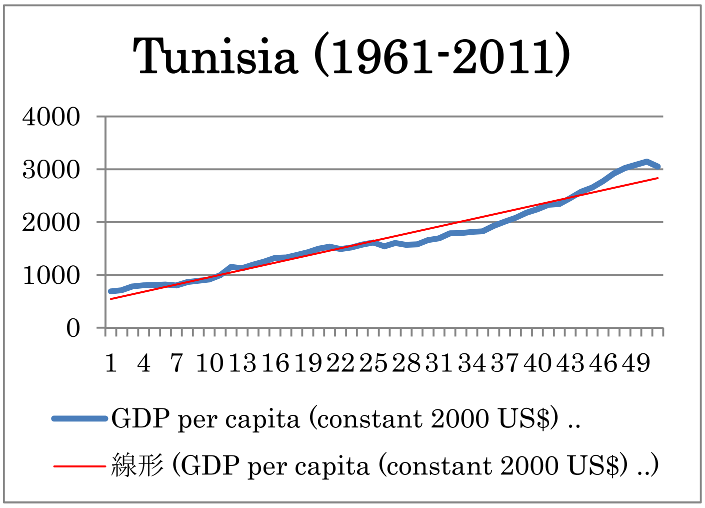
Tables: Bifurcation Cases (Africa)

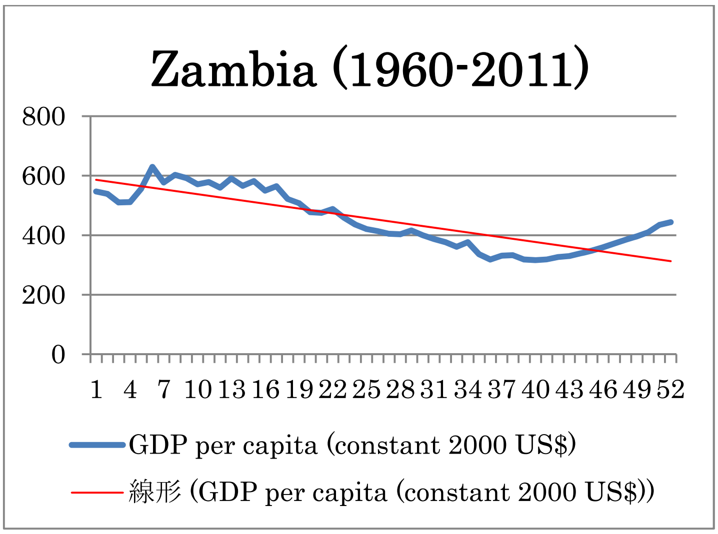












(Source: World Development Indicators)