The Equilibrium Existence and Uniqueness in International Public Two–Good Model

Kenichi Suzuki
Tohoku University

Tatsuyoshi Miyakoshi
Hosei University

2012 January

Abstract

The international public two-good model is an extension of the international public one-good model, allowing for different productivities of producing two public goods across countries. By extending the proof of equilibrium existence, uniqueness and algorithm of Miyakoshi and Suzuki (2011a,b) with one public good, this paper incorporates two public goods into this model and develops an international two public good model, where in general equilibrium is not unique. We give the sufficient condition for the unique equilibrium and explain conditions for multiple equilibriums geometrically.
Keywords: international public two-good model; equilibrium; existence; uniqueness

1 Introduction

Bergstrom and Varian (1986, 1992), who considered the noncooperative Nash equilibrium of voluntary contributions to public goods, prove the existence and uniqueness of the equilibrium, assuming all agents have the same productivity of producing the public good. Many papers extend their seminal work. One extension is to the international public good model, by Ihori (1996, 1999), Boadway and Hayashi (1999), Arce and Sandler (2001), Kim and Shim (2006), Cornes and Hartley (2007), Lei and Vesely (2007), and Miyakoshi and Suzuki (2011a). The main differences between international public good models and public good models are: (i) the 'international public good' is nonrivalrous and nonexcludable across borders, and (ii) each country has a different productivity in producing international public goods. Examples of international public goods are vast forests to prevent global warming, or weapons within the arsenal of the Allies of the North Atlantic Treaty Organization. These differences enable the policy analysis associated with international public good models to be applicable broadly: i.e., global warming or international security.

Another extension is to two public good model, by Bergstrom and Varian (1986), Mutuswami and Winter (2004), Ehlers (2004), Mani and Mukand (2007) and Kung (2008) incorporating two public goods such vast forests and weapons. Recently, Cornes and Hartley (2007) and Miyakoshi and Suzuki (2011a) provided proofs of the existence and uniqueness of equilibrium for the international public good model, extending the proofs of Bergstrom and Varian (1986), Bergstrom and Varian (1992). However, to our knowledge, there are no papers dealing wit the proofs of existence and uniqueness for multiple public good. As a first trial, it is worthy of studying such international public two-good model where any country provides only one public good never provides another public good.

The purpose of the paper is to develop an international public two-good model with proofs of the existence and uniqueness of equilibrium and provide simple applications, extending the proofs of Miyakoshi and Suzuki (2011a,b). Section 2 outlines the international public two-good model. Section 3 provides the proofs of the existence and uniqueness of equilibrium within this model.

2 The Model

Consider a model where there are two public goods, one private good, and n countries ($i = 1, 2, \ldots, n$). Country i consumes an amount x_i of the private good and supplies an amount g_i, h_i of the international public good. The total supply of the public good, G and H, is the sum of g_i, h_i provided by each country. Country i's utility is given by

$$U_i(x_i, G, H) = x_i^{\alpha_i} G^{\beta_i} H^{\gamma_i}$$

where $(\alpha_i, \beta_i, \gamma_i) > 0$ and $\alpha_i + \beta_i + \gamma_i = 1$. Country i has a budget constraint $x_i + p_i g_i + q_i h_i = w_i$, where $w_i > 0$ is the exogenously given national income of country i and $(p_i, q_i) > 0$ is the relative price (cost of production) of public goods in terms of private consumption in country i. A low (high) p_i, q_i means a high (low) productivity in producing the public goods. We also make the Nash assumption that each country believes that the contributions of others are independent of its own. Define the sum of supplies provided by all countries except i by

$$G_{-i} \triangleq \sum_{j \neq i} g_i = G - g_i$$
$$H_{-i} \triangleq \sum_{j \neq i} h_i = H - h_i.$$

Implicitly, each country is choosing not only their contributions, but in fact the equilibrium level of (G, H) itself. When country i contributes to neither of the two public goods, that is $(g_i, h_i) = 0$, it is called a noncontributor. When it makes a positive contribution, it is called a contributor. A contributor belongs to one of the following three sets of countries:

$$
\begin{aligned}
C^G &\triangleq \{i \mid g_i > 0,\ h_i = 0\} \\
C^H &\triangleq \{i \mid g_i = 0,\ h_i > 0\} \\
C^B &\triangleq \{i \mid g_i > 0,\ h_i > 0\}.
\end{aligned}
$$

Denoting C^N the set of noncontributors, these four sets C^G, C^H, C^B, and C^N are mutually exclusive.

Assumption 1
$(p_i, q_i) > 0$ for all i.

Definition 1
A Nash equilibrium in this model is a collection of strategies $\{(x_i, g_i, h_i) \mid i = 1, \ldots, n\}$ such that (x_i, g_i, h_i) is a solution for the following problem for all i:

$$
\left|
\begin{array}{l}
\max\limits_{x_i, g_i, h_i} U_i(x_i, g_i + G_{-i}, h_i + H_{-i}) \\
\quad s.t.\ \ x_i + p_i g_i + q_i h_i = w_i \\
\qquad\quad x_i \geq 0,\ g_i \geq 0,\ h_i \geq 0
\end{array}
\right.
\tag{1}
$$

where $G_{-i} = \sum_{j \neq i} g_j$ and $H_{-i} = \sum_{j \neq i} h_j$.

3 Existence and Uniqueness of the Nash Equilibrium

3.1 Country i's optimal allocation

First, we map (G, H) to (g_i, h_i) assuming the existence of a Nash equilibrium.

Eliminating x_i from problem (1), we obtain an equivalent problem:

$$
\left|
\begin{array}{ll}
\max\limits_{g_i, h_i} & u_i(g_i, h_i; G_{-i}, H_{-i}) \triangleq U_i(w_i - p_i g_i - q_i h_i, g_i + G_{-i}, h_i + H_{-i}) \\
\text{s.t.} & p_i g_i + q_i h_i \leq w_i \\
 & g_i \geq 0,\ h_i \geq 0
\end{array}
\right.
\tag{2}
$$

Given (G_{-i}, H_{-i}), problem (2) is a maximization of a concave function with linear constraints. Its optimal solution (x_i^*, g_i^*, h_i^*) associated with Lagrange multipliers $(\lambda_i, \xi_i, \zeta_i)$ satisfy KKT conditions:

$$p_i g_i^* + q_i h_i^* \le w_i \tag{3a}$$
$$g_i^* \ge 0,\ \ h_i^* \ge 0 \tag{3b}$$
$$\lambda_i \ge 0,\ \ \xi_i \ge 0,\ \ \zeta_i \ge 0 \tag{3c}$$
$$\lambda_i(w_i - p_i g_i^* - q_i h_i^*) = 0 \tag{3d}$$
$$\xi_i g_i^* = 0 \tag{3e}$$
$$\zeta_i h_i^* = 0 \tag{3f}$$
$$u_{ig}(g_i^*, h_i^*; G_{-i}, H_{-i}) - p_i\lambda_i + \xi_i = 0 \tag{3g}$$
$$u_{ih}(g_i^*, h_i^*; G_{-i}, H_{-i}) - q_i\lambda_i + \zeta_i = 0 \tag{3h}$$

where $u_{ig} \triangleq \frac{\partial u_i}{\partial g_i}$ and $u_{ih} \triangleq \frac{\partial u_i}{\partial h_i}$.

For any (G_{-i}, H_{-i}), there exists a feasible allocation (x_i, g_i, h_i) such that $U_i(x_i, G, H) > 0$. Hence at optimum $(x_i^*, G, H) > 0$, since otherwise $U_i(x_i^*, G, H) = 0$. As $x_i^* = w_i - p_i g_i^* - q_i h_i^* > 0$, condition (3d) implies $\lambda_i = 0$.

Introducing $\tilde{\xi}_i \equiv x_i^{1-\alpha} G^{1-\beta} H^{\gamma} \xi_i$ and $\tilde{\zeta}_i \equiv x_i^{1-\alpha} G^{-\beta} H^{1-\gamma} \zeta_i$, the optimal conditions (3b)–(3h) are rewritten as

$$p_i g_i^* + q_i h_i^* < w_i \tag{4a}$$
$$g_i^* \ge 0,\ \ h_i^* \ge 0 \tag{4b}$$
$$\tilde{\xi}_i \ge 0,\ \ \tilde{\zeta}_i \ge 0 \tag{4c}$$
$$\tilde{\xi}_i g_i^* = 0 \tag{4d}$$
$$\tilde{\zeta}_i h_i^* = 0 \tag{4e}$$
$$p_i g_i^* + q_i h_i^* - \tilde{\xi}_i = w_i - p_i \frac{\alpha_i}{\beta_i} G \tag{4f}$$
$$p_i g_i^* + q_i h_i^* - \tilde{\zeta}_i = w_i - q_i \frac{\alpha_i}{\gamma_i} H \tag{4g}$$

From (4c)–(4g), we can derive a correspondence between (g_i, h_i) and (G, H). Let us assume that

$$0 \le G < \frac{\beta_i}{\alpha_i}\frac{w_i}{p_i} \qquad \text{and} \qquad \frac{H}{G} > \frac{q_i}{p_i}\frac{\gamma_i}{\beta_i}\ (\triangleq \delta_i).$$

This is equivalent to

$$p_i g_i^* + q_i h_i^* - \tilde{\xi}_i = \frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i} G > 0 \quad \text{and} \quad \frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i} G > \frac{w_i}{p_i} - \frac{q_i\alpha_i}{p_i\gamma_i} H \tag{5}$$

Inequalities (5) imply $g_i^* > 0$ and $h_i^* = 0$ because, if $h_i^* > 0$, complementarity condition (4e) means $\tilde{\zeta}_i = 0$ and from (4f) and (4g) we have $\frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i} G \le \frac{w_i}{p_i} - \frac{q_i\alpha_i}{p_i\gamma_i} H$, which contradicts the latter inequality of (5).

Similar relationships between (g_i^*, h_i^*) and (G, H) are also derived from (4c)–(4g). We can summarize these conditions as four regions on (G, H)-plane.

Region I) $0 \leq G < \frac{\beta_i}{\alpha_i}\frac{w_i}{p_i}$ and $H = \delta_i G$:

$$p_i g_i^* + q_i h_i^* = w_i - p_i \frac{\alpha_i}{\beta_i} G = w_i - q_i \frac{\alpha_i}{\gamma_i} H \qquad (i \in C^G \cup C^H \cup C^B) \tag{6}$$

Region II) $0 \leq G < \frac{\beta_i}{\alpha_i}\frac{w_i}{p_i}$ and $H > \delta_i G$:

$$g_i^* = \frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i} G \qquad h_i^* = 0 \qquad (i \in C^G) \tag{7}$$

Region III) $0 \leq H < \frac{\gamma_i}{\alpha_i}\frac{w_i}{q_i}$ and $H < \delta_i G$:

$$g_i^* = 0 \qquad h_i^* = \frac{w_i}{p_i} - \frac{\alpha_i}{\gamma_i} H \qquad (i \in C^H) \tag{8}$$

Region IV) $(G, H) \geq (\frac{\beta_i}{\alpha_i}\frac{w_i}{p_i}, \frac{\gamma_i}{\alpha_i}\frac{w_i}{q_i})$:

$$g_i = h_i = 0 \qquad (i \in C^N) \tag{9}$$

The four regions are described in Figure 1. The first orthant of (G, H)–plain is divided into Region II, III, and IV, while the segment OM_i with a thick line indicates Region I. The location of point $M_i = (\frac{\beta_i}{\alpha_i}\frac{w_i}{p_i}, \frac{\gamma_i}{\alpha_i}\frac{w_i}{q_i})$ implies a potential tendency for which type of contributor country i would be, although it also depends on other countries' parameters such as wealth, preferences and relative prices. Roughly saying, the closer M_i gets to the origin, country i is more likely to be a noncontributor. In other words, when country i's wealth is small, its relative preferences of consumption goods to a public good, α_i/β_i and α_i/γ_i, are high, or when its productivity of public goods are low, it tends to be a noncontributor.

Note that only in the case where (G, H) is located in Region I, we can not distinguish country i's type. As will be shown later, this fact is critical for the existence of several equilibriums. When the economy has a single public good, total supply of public good G uniquely determines country i's contribution g_i. This fact ensures the uniqueness of the equilibrium (see Miyakoshi and Suzuki (2011a)). However if the economy has multiple public goods, we cannot rely on such one–to–one correspondence and the results of the single public good economy cannot be generalized directly to a multiple public goods economy.

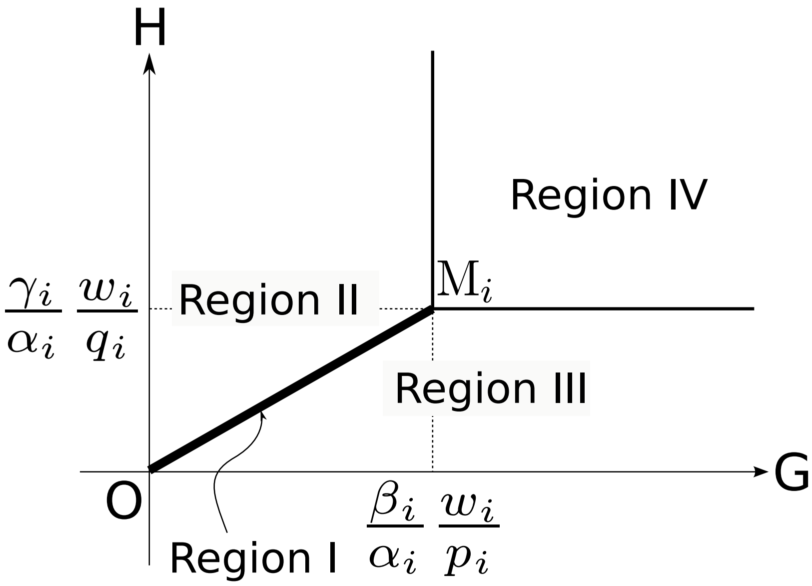

Figure 1: Region I, II, III, and IV on (G, H)–plane

3.2 Existence of an equilibrium

We show the existence of a Nash equilibrium in the two public goods economy.

Theorem 1
A Nash equilibrium exists under Assumption 1.

proof:
Define a set–valued function $\varphi_i : \mathbb{R}^2_+ \to 2^{\mathbb{R}^2_+}$, $i = 1, \ldots, n$.

$$\varphi(G,H) \triangleq \begin{cases} \left\{(g_i,h_i) \mid p_i g_i + q_i h_i = w_i - p_i \dfrac{\alpha_i}{\beta_i} G,\ g_i \geq 0,\ h_i \geq 0\right\} & (G,H) \in \text{ Region I} \\ \left(\dfrac{w_i}{p_i} - \dfrac{\alpha_i}{\beta_i} G,\ 0\right) & (G,H) \in \text{ Region II} \\ \left(0,\ \dfrac{w_i}{q_i} - \dfrac{\alpha_i}{\gamma_i} H\right) & (G,H) \in \text{ Region III} \\ \left(0,\ 0\right) & (G,H) \in \text{ Region IV} \end{cases}$$

From the discussion in Section 3.1, $(g_i, h_i) = \varphi_i(G, H)$ satisfies the optimality condition of country i's problem (2).

Thus we can construct a set–valued function $\Phi : \mathbb{R}^{2n}_+ \to 2^{\mathbb{R}^{2n}_+}$ as

$$\Phi(g_1, \ldots, g_n, h_1, \ldots, h_n) = [\varphi_1(\sum_{i=1}^n g_i, \sum_{i=1}^n h_i), \ldots, \varphi_n(\sum_{i=1}^n g_i, \sum_{i=1}^n h_i)] \quad (10)$$

Φ is closed and for each $v = (g_1, \dots, g_n, h_1, \dots, h_n)$ $\Phi(v)$ is non–empty and convex. Therefore by Kakutani's fixed point theorem, there exists a fixed point, which is a Nash equilibrium. ■

3.3 Uniqueness

In an economy with a single public good, a Nash equilibrium is unique. However in an economy with multiple public goods, it is not always true. Let's see numerical examples. We calculated a Nash equilibrium in the economy of 10 countries with parameters listed in Table 1. Three distinct equilibriums A, B and C were found (Table 2).

i	α_i	β_i	γ_i	w_i	p_i	q_i
1	0.1	0.33	0.57	13.9	0.88	1.84
2	0.1	0.32	0.58	10.1	0.85	1.82
3	0.1	0.49	0.41	16.3	1.14	1.12
4	0.1	0.37	0.53	18.6	0.88	1.55
5	0.1	0.37	0.53	27.2	1.05	1.80
6	0.1	0.42	0.48	21.1	1.04	1.45
7	0.1	0.44	0.46	17.1	1.03	1.29
8	0.1	0.31	0.59	28.8	1.08	2.49
9	0.1	0.49	0.41	22.7	0.95	0.95
10	0.1	0.36	0.54	20.5	1.20	2.11

Table 1: Parameters of 10 countries

First let us compare equilibriums A and B. At Equilibrium A, country 1 provides $(g_1, h_1) = (4.28, 0)$ and $(g_1, h_1) = (0, 2.03)$ at Equilibrium B. on the other hand, country 2 does not change its expenditure. Types of contributions are as follows:

Equilibrium A $\quad C^B = \{6\} \quad C^G = \{1,2,3,7,9\} \quad C^H = \{4,5,8,10\} \quad C^N = \emptyset$

Equilibrium B $\quad C^B = \{7,9\} \quad C^G = \{2,3,6\} \quad C^H = \{1,4,5,8,10\} \quad C^N = \emptyset$

In actuality these two equilibriums are different with respect to our definition of a Nash equilibrium, as shown in the 4th and 7th columns in Table 2, every country's utility level stays the same. This is because i) amount of consumption x_i is unchanged and ii) total supply of public goods, G and H, are identical in both equilibriums.

Next, let's look at Equilibrium C. It is distinguished from the other two equilibriums not only because it has a different pattern of provision

Equilibrium C $\quad C^B = \{5\} \quad C^G = \{1,4,,8,10\} \quad C^H = \{3,6,7,9\} \quad C^N = \{2\}$

	equilibrium A				equilibrium B				equilibrium C			
i	x_i	g_i	h_i	utility	x_i	g_i	h_i	utility	x_i	g_i	h_i	utility
1	10.13	4.28	0.00	29.85	10.13	0.00	2.03	29.85	11.36	2.88	0.00	33.47
2	9.91	0.17	0.00	29.76	9.91	0.17	0.00	29.76	10.05	0.00	0.00	33.04
3	8.70	6.68	0.00	30.31	8.70	6.68	0.00	30.31	9.76	0.00	5.82	33.98
4	9.12	0.00	6.12	29.75	9.12	0.00	6.12	29.75	10.23	9.47	0.00	33.36
5	10.74	0.00	9.13	30.27	10.74	0.00	9.13	30.27	12.04	11.69	1.57	33.94
6	9.47	2.48	6.20	30.14	9.47	11.11	0.00	30.14	10.62	0.00	7.19	33.79
7	8.84	8.00	0.00	30.06	8.84	7.17	0.66	30.06	9.91	0.00	5.54	33.70
8	13.27	0.00	6.24	30.56	13.27	0.00	6.24	30.56	14.87	12.88	0.00	34.27
9	7.32	16.20	0.00	29.77	7.32	12.67	3.51	29.77	8.21	0.00	15.20	33.38
10	12.42	0.00	3.80	30.67	12.42	0.00	3.80	30.67	13.93	5.46	0.00	34.39
		G	H			G	H			G	H	
		37.80	31.50			37.80	31.50			42.39	35.32	

Table 2: Equilibriums A, B, and C

but also because it differs with respect to x_i and (G, H). As a result, at Equilibrium C, all country's utilities are larger than when at Equilibrium A and B.

These observations lead us to the subsequent questions: What is a sufficient condition for an economy to achieve a unique equilibrium? How do we characterize non–unique equilibriums? We will discuss this in the following sections.

3.4 Sufficient Condition for a Unique Equilibrium

Proposition 2
If $\delta_i (= \frac{\gamma_i}{\beta_i}\frac{p_i}{q_i})$ are different for all i, that is,

$$\delta_i \neq \delta_j, \qquad \text{for all } i, j, \tag{11}$$

there exists only one pair of equilibrium supply (G, H).

proof:

Let us suppose that condition (11) holds and there exist different equilibriums with (G^*, H^*) and $(\bar{G}, \bar{H})$. Denote C^{*B}, C^{*G}, C^{*H}, and C^{*N} types of contributor in equilibrium with (G^*, H^*) and denote $\bar{C}^B$. $\bar{C}^G$, $\bar{C}^H$, and $\bar{C}^N$ types of contributor in equilibrium with (G^*, H^*). Without the loss of generality, we assume $G^* > \bar{G}$. Thus we consider three cases: i) $H^*/G^* < \bar{H}/\bar{G}$, ii) $H^*/G^* > \bar{H}/\bar{G}$, and iii) $H^*/G^* = \bar{H}/\bar{G}$.

i) $H^*/G^* < \bar{H}/\bar{G}$:

In this case, by a change of total supply from (G^*, H^*) to $(\bar{G}, \bar{H})$, country $i \in C^{*G}$ remains in $\bar{C}^G$. We can say $C^{*G} \subset \bar{C}^G$ and for $i \in C^{*G}$, we have

$$g_i^* = \frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i} G^* > \frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i} \bar{G} = \bar{g}_i$$

As $\bar{G} = \sum_{i \in \bar{C}^G} \bar{g}_i$,

$$\bar{G} \geq \sum_{i \in C^{*G}} \bar{g}_i > \sum_{i \in C^{*G}} g_i^* = G^*.$$

This then contradicts the assumption $G^* > \bar{G}$.

ii) $H^*/G^* > \bar{H}/\bar{G}$:

From assumption $G^* > \bar{G}$, $H^*/G^* > \bar{H}/\bar{G}$ implies $H^* > \frac{G^*}{\bar{G}}\bar{H} > \bar{H}$.

In a similar way to case i), we have $C^{*H} \subset \bar{C}^H$. For $i \in C^{*H}$, we have

$$h_i^* = \frac{w_i}{q_i} - \frac{\alpha_i}{\gamma_i} H^* > \frac{w_i}{q_i} - \frac{\alpha_i}{\gamma_i} \bar{H} = \bar{h}_i$$

Again we can derive an inequality $H^* < \bar{H}$, which contradicts to the assumption.

iii) $H^*/G^* = \bar{H}/\bar{G}$:

Almost the same argument can be applied here, though we should take into account the possibility that both (G^*, H^*) and $(\bar{G}, \bar{H})$ exist in one country's Region I. Suppose $\delta_k = H^*/G^*$, which means country k can be any type of contributor. However, even in this case, except k, all the countries in $C^{*G} \cup C^{*H}$ are included in $\bar{C}^G \cup \bar{C}^H$. It can be shown that any shift of k's type increase either $\bar{G}$ or $\bar{H}$. Therefore we can also derive contradiction. ■

An essential point in Proposition 2 is that if condition (11) holds, then at most one country belongs to C^B. When $C^B = \emptyset$, all contributors are members of either C^G or C^H and the equilibrium is reduced to the similar structure of a single public good economy.

Now suppose we know C^G and C^H and $C^B = \emptyset$. Applying the algorithm for finding the equilibrium supply in a single public good economy, we can derive the equilibrium supply:

$$G = \sum_{i \in C^G} \frac{w_i}{p_i} / \left(1 + \sum_{i \in C^G} \frac{\alpha_i}{\beta_i} \right) \tag{12}$$

$$H = \sum_{i \in C^H} \frac{w_i}{q_i} / \left(1 + \sum_{i \in C^H} \frac{\alpha_i}{\gamma_i} \right) \tag{13}$$

and contributor i's gifts (g_i^*, h_i^*) are given by (7) and (8). See Miyakoshi and Suzuki (2011a) for details.

Next consider the case that only one country provides both public goods, say $C^B = \{k\}$. Each contributors' gifts are written as

$$\begin{aligned} g_i^* &= \frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i} G, & i \in C^G \\ h_i^* &= \frac{w_i}{q_i} - \frac{\alpha_i}{\gamma_i} H, & i \in C^H \\ p_k g_k^* + q_k h_k^* &= w_k - p_k \frac{\alpha_k}{\beta_k} G \end{aligned}$$

As $C^B = \{k\}$, $H = \delta_k G$. Using the definition of G and H, this becomes

$$G = \sum_{i \in C^G} g_i + g_k \qquad H = \sum_{i \in C^H} h_i + h_k,$$

and substituting H by $\delta_k G$, we have

$$\begin{aligned} G &= \frac{w_k + p_k \sum_{i \in C^G} w_i/p_i + q_k \sum_{i \in C^H} w_i/q_i}{p_k \alpha_k/\beta_k + p_k \left(1 + \sum_{i \in C^G} \alpha_i/\beta_i\right) + q_k \delta_k \left(1 + \sum_{i \in C^H} \alpha_i/\gamma_i\right)} \\ H &= \delta_k G. \end{aligned}$$

When one or no country exists in C^B, each country's contribution is uniquely associated with total supply (G, H). Combining this fact with proposition 2, we can claim the uniqueness of a Nash equilibrium.

Theorem 3
If condition (11) *holds, a Nash equilibrium is unique.*

3.5 Multiple Equilibriums

As shown by the example in section 3.3, a two public goods economy may have more than one equilibrium. In this section, we focus on the characteristic of multiple equilibriums.

The sufficient condition for an unique equilibrium presented in the previous section implies that $|C^B|$, the cardinality of C^B, is less than one. This suggests that if $|C^B| \geq 2$, we can find distinct equilibriums. Actually Equilibrium B of the example has $C^B = \{7, 9\}$. We can formally state a necessary condition for multiple equilibriums as follows.

Proposition 4
Let us suppose that one equilibrium has a total supply pair $(\bar{G}, \bar{H})$. Let δ denote the ratio $\bar{H}/\bar{G}$. If there exists another equilibrium, the following conditions hold.

i) Let $\bar{C}^B$ be a union of C^B in all equilibriums, $\bar{C}^B = \cup C^B$. Then $|\bar{C}^B| \geq 2$ and for all $i \in \bar{C}^B$,

$$\delta_i = \frac{\gamma_i}{\beta_i}\frac{p_i}{q_i} = \delta \tag{14}$$

ii) There exists interval $I^G \triangleq [G^{\min}, G^{\max}]$ such that for any $G \in I^G$, $(G, H) = (G, \delta G)$ is an equilibrium pair.

proof:

i) Contrapositive to the statement that $|C^B| \leq 1$ implies the uniqueness of the equilibrium, is that multiple equilibriums have at least one C^B containing no less than two members.

All the equilibrium supply pairs (G, H) are located on the lay from origin on (G, H)-plane, otherwise we can derive contradictions in the same manner as in the proof of Proposition 2. For all $i \in C^B$, equation (6) implies

$$\delta_i = \frac{\gamma_i}{\beta_i}\frac{p_i}{q_i} = \frac{H}{G} \quad \text{for all } i \in C^B. \tag{15}$$

Hence the direction of the lay is $(1, \delta)$ and (14) holds for all $i \in \bar{C}^B$.

ii) Suppose there is an equilibrium with (G, H) and let $\delta = H/G$. Define sets $\hat{C}^B$, $\hat{C}^G$, and $\hat{C}^H$ as

$$\hat{C}^B = \{i \mid \delta_i = \delta\} \quad \hat{C}^G = \{i \mid \delta_i > \delta\} \quad \hat{C}^H = \{i \mid \delta_i < \delta\}. \tag{16}$$

In this economy, every equilibrium is a solution of the system:

$$\begin{cases} p_i g_i + q_i h_i = \max\{0,\ w_i - p_i \dfrac{\alpha_i}{\beta_i} G\} & i \in \hat{C}^B \\ p_i g_i = \max\{0,\ w_i - p_i \dfrac{\alpha_i}{\beta_i} G\} & i \in \hat{C}^G \\ q_i h_i = \max\{0,\ w_i - q_i \dfrac{\alpha_i}{\gamma_i} \delta G\} & i \in \hat{C}^H \\ \displaystyle\sum_{i \in \hat{C}^B \cup \hat{C}^G} g_i = G, \quad \sum_{i \in \hat{C}^B \cup \hat{C}^H} h_i = \delta G \\ g_i \geq 0, \quad h_i \geq 0, \quad i \in \hat{C}^B \cup \hat{C}^G \cup \hat{C}^H \end{cases} \tag{17}$$

Consider maximizing G over the system (17). Since (17) is a closed nonempty set, this problem displays the maximum. It also displays the minimum of G.

Suppose that we are at $(G^{\max}, \delta^{\max})$. The equilibrium is a solution of linear system:

$$\begin{cases} p_i g_i + q_i h_i = w_i - p_i \dfrac{\alpha_i}{\beta_i} G & i \in C^B \\ p_i g_i = w_i - p_i \dfrac{\alpha_i}{\beta_i} G & i \in C^G \\ q_i h_i = w_i - q_i \dfrac{\alpha_i}{\gamma_i} \delta G & i \in C^H \\ \displaystyle\sum_{i \in C^B \cup C^G} g_i = G, \quad \sum_{i \in C^B \cup C^H} h_i = \delta G \\ g_i \geq 0, \quad h_i \geq 0, \quad i \in C^B \cup C^G \cup C^H \end{cases} \tag{18}$$

As linear system (18) represents a polyhedron with nonempty interior, thus $G < G^{\max}$ can be a solution. Decreasing G, $i \in C^N$ may be added to C^B, C^G, or C^H to satisfy (17). Addition of a new equation will expand the system, allowing for a further decrease of G. By repeating this process, we finally reach $G^{\min}$. ■

Figure 2 illustrates situations described in Proposition 4. The thick blue line represents the segment consisting of equilibrium pair (G, H). M_i $(i = 1, \ldots, 10)$ indicates the threshold points of 10 countries. That means, if $M_i \geq (G, H)$, country i is a non-contributor. M_2 is in the middle of the segment, implying that country i shifts from a contributor to a non contributor when G increases.

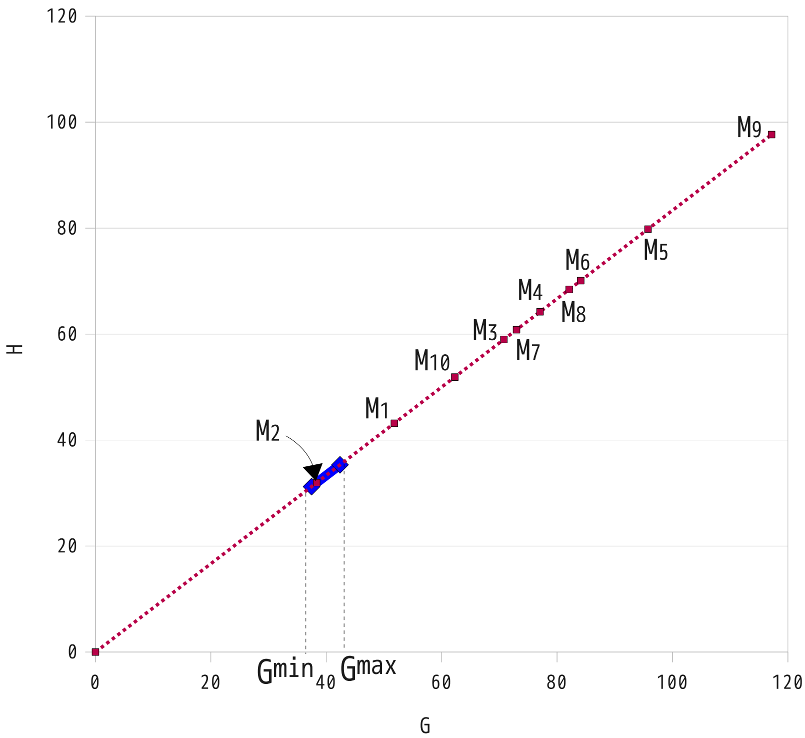

Figure 2: Segment of equilibrium pair (G, H) (thick blue line)

It may be unlikely that some countries share the same δ as described in Figure 2, when prices (p_i, q_i) and preferences (β_i, γ_i) vary across countries. If we consider a group of consumers with homogeneous preferences in a domestic economy, they have a common δ. However it oversimplifies the situation.

Proposition 4 gives us a geometric interpretation of multiple equilibriums, yet it is not clear what characteristics each equilibrium possesses. We can know few things about an arbitrary equilibrium in the middle of the segment, since it is just a point in the polyhedron defined by (18). Therefore we focus on the special point $(G^{\max}, \delta G^{\max})$. To do so, we must introduce two types of equilibriums.

Definition 2

1. *If an equilibrium is achieved with $C^B = \emptyset$, it is a separating equilibrium.*

2. *If an equilibrium is achieved with C^B consisting of a single country, it is a semi–separating equilibrium.*

Note that a unique equilibrium under condition (11) is either a separating or a semi–separating equilibrium. Without condition (11), a semi–separating equilibrium is key to establishing uniqueness .

Theorem 5
Assume that $(p_i, q_i) \neq (p_j, q_j)$ for all i, j. Then the equilibrium with a total supply $(G^{\max}, \delta G^{\max})$ is a unique semi–separating equilibrium. Moreover at this equilibrium all the countries' utilities are larger than at another equilibriums.

proof:
Suppose we are given C^B, C^G, and C^H at equilibrium with $(G^{\max}, \delta G^{\max})$. Then $G^{\max}$ is an optimal solution of a linear programming problem:

$$\max\{G \mid (G, g_1, \cdots, g_n, h_1, \cdots, h_n) \text{ satisfies (18)}\}. \tag{19}$$

Let m be the total number of countries in C^B, C^G, and C^H in linear system (18), then it has $2m+1$ variables and $m+2$ equations. We can verify that the system is non degenerate under the assumption of price. From the theory of LP (see Chvátal (1983) for example), its basis consists of $m+2$ variables and either g_i or h_i, as well as G, must be basic variables. Therefore only one country k's contribution (g_k, h_k) enters the basis as a rank of basis matrix is $m+2$. In other words, every basic feasible solution corresponds to a semi–separating equilibrium. We can show that at optimum all non basic variables have negative reduced costs, which implies that the optimal (basic) solution is unique.

Country i's consumption x_i, written as

$$x_i = w_i - p_i g_i - q_i h_i = \max\left\{w_i, \min\left\{p_i \frac{\alpha_i}{\beta_i} G, q_i \frac{\alpha_i}{\gamma_i} H\}\right\}\right\},$$

is a nondecreasing function of (G, H). All the country's utility $U(x_i, G, H)$ increases as (G, H) increases. As a result $(G^{\max}, \delta G^{\max})$ dominates all other equilibriums. ■

Remark I: The equilibrium with total supply $(G^{\min}, \delta G^{\min})$ is also unique for the same reason.
Remark II: If price (p_i, q_i) are common to all countries, the equilibrium with total supply $(G^{\max}, \delta G^{\max})$ is a unique separating equilibrium. This is because the linear system (18) is degenerating and the rank of the basis matrix equals $m+1$.

4 Conclusion

Quite a few papers have been discussed properties of a single public good economy. Among them important results are the uniqueness of a Nash equilibrium and the neutrality to income transfer. Although a multiple public goods model can be constructed as a simple extension of a single public good model, behaviors of the equilibrium are significantly different.

One major distinction is plurality of Nash equilibriums. When more than two countries have same $\delta_i = H/G$, in other words, when these countries' M_i and (G, H) are located on the same line from origin, one can find several equilibriums in which each country's utility varies according to various values of (G, H). We introduced a separating and a semi–separating equilibrium and showed the sufficient condition to achieve a unique equilibrium. In addition, we proved that there exists a unique semi–separating equilibrium among the set of equilibriums which dominates other equilibriums.

The two public goods economy is well characterized by the semi separating equilibrium where only one member belongs to C^B. So a natural question is: "who spends for both public goods?" Although we can not fully answer to it, it can be said that M_i, $i \in C^B$, is likely to exists at the northeast area in G-H plane far from origin. It implies, country $i \in C^B$, would have a large wealth, higher productivities of both goods and an adequate preference ratio $\beta_i : \gamma_i$. Therefore the semi–separating equilibrium suggests the relationship of the superpower and other countries in the real world. Further analysis will be needed for more suggesting implications of the two public goods economy.

References

ARCE, M. AND T. SANDLER (2001): "Transnational public goods: strategies and institutions," *European Journal of Political Economy*, 17, 493–516.

BERGSTROM, T., B. L. AND H. VARIAN (1986): "On the private provision of public good," *Journal of Public Economics*, 29, 25–49.

——— (1992): "Uniqueness of Nash equilibrium in private provision of public goods: an improved proof," *Journal of Public Economics*, 49, 391–392.

BOADWAY, R. AND M. HAYASHI (1999): "Country size and the voluntary provision of international public goods," *European Journal of Political Economy*, 15, 619–638.

CHVÁTAL, V. (1983): *Linear Programming*, Freeman and Co., New York.

CORNES, R. AND R. HARTLEY (2007): "Aggregative Public Good Games," *Journal of Public Economic Theory*, 9, 201–219.

EHLERS, L. (2004): "Multiple public goods and lexicographic preferences: replacement principle," *Journal of Mathematical Economics*, 37, 1–15.

Ihori, T. (1996): "International public goods and contribution productivity differentials," *Journal of Public Economics*, 61, 139–154.

——— (1999): "An economic analysis of public Transfers," *Japanese Economic Review*, 50, 44–60.

Kim, J. and S. Shim (2006): "Incentive mechanisms for international public goods under uncertainty of production costs," *Economics Letters*, 92, 311–316.

Kung, F.-C. (2008): "Voluntary contributions to multiple public goods in a production economy with widespread externalities," *Journal of Mathematical Economics*, 44, 1364–1378.

Lei, V., T. S. and F. Vesely (2007): "Foreign aid and weakest-link international public goods: an experimental study," *European Economic Review*, 51, 599–623.

Mani, A. and S. Mukand (2007): "Democracy, visibility and public good provision," *Journal of Development Economics*, 83, 506–529.

Miyakoshi, T. and K. Suzuki (2011a): "The existence and uniqueness of equilibrium in the international public good model," *Applied Economics Letters*, 18, 1751–1754.

——— (2011b): "A Practical Algorithm for Identifying Free Riders in International Public Goods: the Free Rider Problem and ODA methods," Mimeo, Osaka University.

Mutuswami, S. and E. Winter (2004): "Efficient mechanisms for multiple public goods," *Journal of Public Economics*, 88, 629– 644.

Warr, P. (1983): "The private provision of a public good is independent of the distribution of income," *Economics Letters*, 13, 207–211.